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Maximum Weight Relaxed Cliques and Russian Doll Search Revisited

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Abstract

Trukhanov et al. [Trukhanov S, Balasubramaniam C, Balasundaram B, Butenko S (2013) Algorithms for detecting optimal hereditary structures in graphs, with application to clique relaxations. Comp. Opt. and Appl., 56(1), 113–130] used the Russian Doll Search (RDS) principle to effectively find maximum hereditary structures in graphs. Prominent examples of such hereditary structures are cliques and some clique relaxations intensely discussed and studied in network analysis. The effectiveness of the tailored RDS by Trukhanov et al. for s-plex and s-defective clique can be attributed to their cleverly designed incremental verification procedures used to distinguish feasible from infeasible structures. In this short note, we clarify the incompletely presented verification procedure for s-plex and present a new and simpler incremental verification procedure for s-defective cliques with a better worst-case runtime. Furthermore, we develop an incremental verification for s-bundle, giving rise to the first exact algorithm for solving the maximum cardinality and maximum weight s-bundle problems.

Key words: Relaxed clique, Russian Doll Search, Optimal hereditary structures, Maximum weight Π problem

1. Introduction

The combinatorial branch-and-bound by Östergård (2002) is among the most powerful exact algorithms to identify maximum cardinality and maximum weight cliques. It follows the *Russian Doll Search* (RDS) principle originally introduced by Verfaillie *et al.* (1996) for solving valued constraint satisfaction problems. In the context of graph theory, it is applicable to find optimal hereditary structures. In particular, Trukhanov *et al.* (2013) solve maximum cardinality s-plex and s-defective clique problems. These are examples of relaxed cliques, which are hereditary and of interest in social network analysis (see Pattillo *et al.*, 2013; Fortunato, 2010).

Let $G = (V, E)$ be a simple graph with finite vertex set V and edge set E. For any subset $S \subseteq V$, the vertex-induced subgraph of S is $G[S] = (S, E \cap (S \times S))$. A graph property Π is *hereditary* on induced subgraphs if for any subset $S \subseteq V$ with $G[S]$ satisfying property Π, any subset $S' \subset S, S' \neq \emptyset$ induces a subgraph $G[S']$ that satisfies Π. A property Π is *nontrivial* if it is true for all $G[S]$ induced by singleton sets $S = \{i\}, i \in V$ and not satisfied by every graph. A property Π is *interesting* if there exist graphs G of arbitrary size satisfying Π. Yannakakis (1978) has shown that the determination of a maximum cardinality set S satisfying Π, i.e., the *maximum cardinality* Π *problem* is $\mathcal{NP}$-hard for Π that are hereditary, nontrivial, and interesting. In the following we refer to these properties as the Yannakakis properties. For given vertex weights $w_i, i \in V$, the *maximum weight* Π *problem* seeks for a set S with maximum weight $w(S) = \sum_{i \in S} w_i$ satisfying Π. For hereditary Π, the weights can be assumed to be non-negative because otherwise the corresponding vertex can never be in an optimal solution.

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One prominent example for a structure that satisfies the Yannakakis properties is the clique: A set $S \subseteq V$ is a *clique* if $G[S]$ is complete, i.e., all vertices are adjacent. Pattillo *et al.* (2013) show that first-order clique relaxations can be derived from relaxing the distance, degree, density, or connectivity requirements of cliques. Note that cliques are perfect in the sense that they have maximum density, their vertices have maximum degree, and pairs of vertices have minimum distance and maximum connectivity. In the following, we formally introduce these graph parameters and define those relaxed cliques that satisfy the Yannakakis properties.

For $i, j \in V$, $\text{dist}_G(i, j)$ is the minimum length of a path in G connecting i and j. For $s \geq 1$, $S \subseteq V$ is an s-clique if $\text{dist}_G(i, j) \leq s$ for all $i, j \in S$. As every s-clique is an ordinary clique in the sth power graph of G, the search for maximum s-cliques can be performed with any maximum clique algorithm. Therefore, we do not consider s-cliques in the remainder of the paper.

Let $i \in V$ be any vertex and let $S \subseteq V$ be any subset of vertices. Vertices adjacent to i are denoted by $N(i)$. The *vertex degree* in G of vertex i is $|N(i)|$ and is denoted by $\deg_G(i)$. The *minimum vertex degree* of G is $\delta(G) = \min_{i \in V} \deg_G(i)$. For $s \geq 1$, $S \subseteq V$ is an *s-plex* if $\delta(G[S]) \geq |S| - s$.

The set $E(S)$ is the set of edges in G with both endpoints in S. For $s \geq 0$, S is an *s-defective clique* if $|E(S)| \geq \binom{|S|}{2} - s$.

A set $C \subset V$ is a *vertex cut* of a connected graph $G = (V, E)$ if $G[V \setminus C]$ is a disconnected graph. Note that any vertex cut C has at most $|V| - 2$ elements. The vertex connectivity $\kappa(G)$ of G is the size of a minimum cardinality vertex cut. For cliques S, $G[S]$ does not have any vertex cuts, and therefore one defines $\kappa(G[S]) = |S| - 1$. A graph is called k-vertex-connected if its vertex connectivity is k or greater. The local connectivity $\kappa_G(i, j)$ of two different and non-adjacent vertices $i, j \in V$ is the minimum size of a vertex cut C disconnecting i and j in $G[V \setminus C]$. For adjacent vertices i and j, one defines $\kappa_G(i, j) = \infty$. Then, if G is not a clique, $\kappa(G)$ equals the minimum of $\kappa_G(i, j)$ over all pairs of different vertices $i, j \in V$. Two i-j-paths are called vertex-disjoint if they have no vertices in common except i and j. Menger's theorem (Menger, 1927) states that the minimum size of a vertex cut disconnecting i and j is equal to the maximum number of vertex-disjoint paths connecting i and j. Hence, for non-adjacent vertices i and j, $\kappa_G(i, j)$ is the maximum number of vertex-disjoint i-j-paths. For $s \geq 1$, S is an *s-bundle* if $\kappa(G[S]) \geq |S| - s$.

Note that any (ordinary) clique S is a 1-plex, 0-defective clique, and 1-bundle. For $s > 1$, every $(s-1)$-defective clique and every s-bundle is an s-plex, but the reverse is generally not true.

A prerequisite of RDS is that the n vertices V are ordered into a sequence $(v_1, v_2, \ldots, v_n)$. Instead of one depth-first branch-and-bound search, RDS performs n searches. Starting from $i = n$, the ith search determines a maximum weight Π set for $G[\{v_i, v_{i+1}, \ldots, v_n\}]$ with the initial set $S = \{v_i\}$. In every iteration, i is decreased by 1 so that a sequence of lower bounds $LB_n, LB_{n-1}, \ldots, LB_2, LB_1$ is computed. These bounds allow an improved pruning compared to single branch-and-bound searches (see Section 2). At each stage of the RDS search, the current solution P satisfies Π. Moreover, a set of candidates C with $P \cup \{c\}$ satisfies Π for all $c \in C$ is maintained. Whenever P is enlarged, C has to be adjusted, i.e., candidate vertices not compatible with the new set P are removed from C. The test whether $P \cup \{c\}$ for a candidate vertex $c \in C$ satisfies Π is called the verification procedure.

Trukhanov *et al.* (2013) presented straightforward and incremental verification procedures for s-plex and s-defective clique. While straightforward procedures are simpler to implement, the incremental verification procedures have a better runtime complexity.

The contribution of this paper is threefold: First, we clarify the incremental verification procedure for s-plex because the description in (Trukhanov *et al.*, 2013) is incomplete. Second, we present a new and simpler incremental verification procedure for s-defective cliques with a better worst-case runtime complexity. Third, no solution algorithm for s-bundle neither heuristic nor exact has been presented in the literature. We develop an incremental verification procedure and herewith introduce a first, RDS-based algorithm for maximum-weight s-bundle.

The remainder of the paper is structured as follows: In Section 2, we briefly summarize RDS and present the new incremental verification procedures. In Section 3, the effectiveness of the new RDS algorithms is analyzed in a computational study. Final conclusions are drawn in Section 4.

2. Russian doll search

Algorithm 1 presents RDS for the maximum weight Π problem in a slightly modified version compared to (Trukhanov *et al.*, 2013). Different strategies for the vertex ordering in Step 1 were discuss and analyzed by Trukhanov *et al.* (2013). For unit weights, a degree based ordering as suggested by (Carraghan and Pardalos, 1990) turned out to give the best overall performance for RDS. Herein, v_n is first chosen as a minimum degree vertex. Then, iteratively from $i = n-1$ down to 1, the vertex v_i is selected such that v_i has minimum degree in $G[V \setminus \{v_{i+1}, \dots, v_n\}]$.

RDS maintains $n+1$ lower bounds: The global bound LB (Step 2) is the weight of the best solution found so far. Moreover, the n branch-and-bound searches are initiated in the main loop (Steps 3 to 6). Each search produces a best solution of weight LB_i by calling the procedure FindMax which perform the actual branch-and-bound on $G[\{v_i, \dots, v_n\}]$. The initial candidate set C is computed in Step 4 using a problem specific verification procedure.

Algorithm 1: Russion Doll Search (RDS) for the Maximum Weight Π Problem

Input: Vertex-weighted graph $G = (V, E, w_i)$ and property Π
Output: Vertex set S inducing a maximum weight Π subgraph $G[S]$

1 Order vertices $(v_1, v_2, \dots, v_n)$
2 Set $LB := 0$ and $S := \emptyset$
3 **for** $i := n, n-1, \dots, 1$ **do**
4   Set $C := \{v_j : j > i, \{v_i, v_j\}$ satisfies $\Pi\}$ `// Π-verification`
5   Call FindMax(C, $\{v_i\}$)
6     $LB_i := LB$

The Procedure FindMax is called with the candidate set C and the current set P. RDS always keeps C and P such that $P \cup \{v\}$ satisfies Π for each $v \in C$ (in the following referred to as *consistency*). Therefore, if C is empty (Steps 1 to 4), an inclusion maximal solution is found and tested for optimality.

Procedure FindMax(C, P)

Input: Candidate set C and current set P

1 **if** $C = \emptyset$ **then**
2   **if** $w(P) > LB$ **then**
3   Set $LB := w(P)$ and $S := P$
4   **return**
5 **while** $C \neq \emptyset$ **do**
6   **if** $w(C) + w(P) \leq LB$ **then return** `// Pruning 1`
7   Set $i := \min\{j : v_j \in C\}$
8   **if** $LB_i + w(P) \leq LB$ **then return** `// Pruning 2`
9   Set $C := C \setminus \{v_i\}$ and $P' := P \cup \{v_i\}$
10   PrepareAuxiliaryInformation(C, P')
11   Set $C' := \{v \in C : P' \cup \{v\}$ satisfies $\Pi\}$ `// Π-verification`
12   Call FindMax(C', P')

The main loop of the FindMax procedure is presented in the Steps 5 to 12. In every iteration, two tests are performed in order to prune the search. The first pruning test in Step 6 only utilizes the weights of C and P. This is a standard test in any branch-and-bound. The more sophisticated pruning test (Step 8) refers to the candidate vertex v_i with the smallest index (in the vertex ordering). This is the candidate vertex providing the best lower bound LB_i available from preceding branch-and-bound searches. Trukhanov *et al.* (2013) stress that a good vertex ordering is one that encourages the pruning in Step 8. The effectiveness of RDS can be attributed to this second pruning step. Note that in both pruning steps the conditions are sharpened compared to Trukhanov *et al.* (2013) who use less than instead of less than or equal.

If no pruning is possible, a new current set P' and a new candidate set C' are computed (Steps 9 and 11). The vertex v_i (from the pruning step) is added to the current set and removed from the candidate set. In order to maintain consistency, the verification in Step 11 checks the same current set together with each candidate $v \in C$ one by one. It is crucial for the performance of RDS that this verification is fast.

We have added the Step 10 intended to accelerate the verification procedure. While straightforward verification performs an ad hoc test, incremental verification procedures rely on the a priori computation of some auxiliary information. For example, one can exploit that P' is identical in each call to the verification procedure. We present the problem-specific incremental verification procedures of Trukhanov *et al.* (2013) and ours in the following sections.

Finally, the recursion in Step 12 continues the search with the updated candidate set C' and the updated current set P' maintaining consistency.

2.1. s-Plex

A straightforward verification for s-plex and $S = P' \cup \{v\}$ computes the vertex degree of each vertex of $G[S]$. This takes quadratic time $\mathcal{O}(|P'|^2)$.

The incremental verification procedure presented by Trukhanov *et al.* (2013) uses the following information: For each vertex $v \in P'$, $\textsc{nncnt}[v]$ is the number of non-adjacent vertices ("non-neighbors") in $G[P']$. The update of $\textsc{nncnt}[v]$ can use the fact that P' grows by exactly one vertex from one recursive call of FindMax to the next. If P' has been extended by vertex v_i, i.e., $P' = P \cup \{v_i\}$ (see Step 9), $\textsc{nncnt}[v]$ increases by 1 for non-adjacent v_i and v. The values $\textsc{nncnt}[v]$ for adjacent vertices v_i and v remain identical. This update requires $\mathcal{O}(|P'|)$ time.

A vertex $v \in P'$ is saturated in an s-plex P' if it has smallest possible degree $\deg_{G[P']}(v) = |P'| - s$. The set SAT consists of those vertices in P' which become saturated in the current call to `FindMax`. Vertices already saturated in $P = P' \setminus \{v_i\}$ are not in SAT. The presented verification procedure for $P' \cup \{v\}$ (for $v \in C$) checks only for vertices u in the saturated set SAT if u and v are adjacent. If all are adjacent $P' \cup \{v\}$ satisfies Π. If not $P' \cup \{v\}$ does not satisfy Π. This test takes $\mathcal{O}(|\mathrm{SAT}|)$ time. Note that the computation of SAT is a byproduct of the update of $\textsc{nncnt}[v]$, which requires $\mathcal{O}(|P'|)$ time.

The presented incremental verification procedure of Trukhanov *et al.* (2013) may incorrectly classify $P' \cup \{v\}$ as a feasible structure satisfying Π. What is missing is the check that v must have degree at least $|P'| - s$ in $G[P' \cup \{v\}]$, i.e., v must have $|P'| - s$ or more adjacent vertices in P'. We would like to stress that Trukhanov *et al.* have implemented a different and apparently correct version of a verification procedure, since their solutions are valid.

We suggest the following modifications for the presentation of a correct and efficient incremental verification procedure: The information about the number $\textsc{nncnt}[v]$ of non-adjacent vertices in $G[P' \cup \{v\}]$ should be provided for vertices $v \in C \cup P'$, i.e., not only for the current set P' but also for the candidate set C. The computation can again be done recursively taking $\mathcal{O}(|P'| + |C|)$ time. The degree of any candidate $v \in C$ in P' is then $|P'| - \textsc{nncnt}[v]$ so that the minimum degree check reduces to $\textsc{nncnt}[v] \leq s$.

Summing up, our modified incremental verification procedure takes the same time $\mathcal{O}(|\mathrm{SAT}|)$ for the actual verification, while the update (Step 10) takes $\mathcal{O}(|P'| + |C|)$ time instead of the $\mathcal{O}(|C|)$ time as presented by Trukhanov *et al.* Since the update takes places only once per recursive call of FindMax, this increased time to compute the auxiliary information is insignificant. Note however that storing and updating $\textsc{nncnt}[v]$ for every recursive call of FindMax consumes $\mathcal{O}(|V|^2)$ memory, which may become prohibitive for large-scale graphs with a huge number of vertices.

2.2. s-Defective clique

A straightforward verification for s-defective clique with $S = P' \cup \{v\}$ computes the number of missing edges in $G[S]$ and takes quadratic time $\mathcal{O}(|P'|^2)$.

Trukhanov *et al.* (2013) exploit the fact that every s-defective clique is an $(s+1)$-plex so that the above incremental verification procedure can be used to quickly reject S not satisfying Π. As discussed above, this consumes $\mathcal{O}(|\mathrm{SAT}|)$ for the verification and $\mathcal{O}(|P'|+|C|)$ for Step 10. If this pre-test identifies $P' \cup \{v\}$ as a feasible $(s+1)$-plex, the straightforward verification is applied leading to a worst-case run time of $\mathcal{O}(|P'|^2)$.

It is possible to design an alternative incremental verification procedure with constant time test and linear time update step. We maintain the information about the number $\textsc{nncnt}[v]$ of non-adjacent vertices in $G[P' \cup \{v\}]$ only for vertices $v \in C$ of the candidate set. Moreover, we count the overall number $\textsc{nnv}$ of non-adjacent vertices in $G[P']$. Testing if $P' \cup \{v\}$ satisfies Π for $v \in C$ now means checking $\textsc{nncnt}[v] + \textsc{nnv} \leq s$. The update of $\textsc{nnv}$ is also constant because the new value, to be computed when the vertex v_i is added to P, is identical to $\textsc{nnv} + \textsc{nncnt}[v_i]$. The update of $\textsc{nncnt}[v]$ for all $c \in C$ runs in $\mathcal{O}(|C|)$.

2.3. s-Bundle

The maximum cardinality and maximum weight s-bundle problem have not been solved before. However, RDS can immediately be adapted to this relaxed clique variant using a straightforward verification procedure. For $S = P' \cup \{v\}$, it computes $\kappa_{G[S]}(i,j)$ for all non-adjacent pairs $i, j \in S$ with $\{i,j\} \notin E$. S is no s-bundle if $\kappa_{G[S]}(i,j) < |S| - s$ for any such pair i and j. Otherwise, S is an s-bundle.

Recall that for any graph $G = (V, E)$ and any integer $k \geq 1$ the condition $\kappa_G(i,j) \geq k$ is equivalent to the existence of k vertex-disjoint paths connecting i and j in G (Menger, 1927). The existence of k vertex-disjoint paths in G is in turn equivalent to the existence of a feasible flow of size k between vertices i^+ and j^- in the following network $\mathcal{N} = (N, A)$ (see, e.g., Kammer and Täubig, 2004). For each vertex $v \in V$, N contains two vertices v^- and v^+, which are connected by the arc $(v^-, v^+) \in A$. Moreover, for each edge $\{v, w\} \in E$, the two arcs (v^+, w^-) and (w^+, v^-) are in A. All arcs of A have unit capacity.

The existence of a flow of size k between vertices i^+ and j^- in $\mathcal{N}$ can be tested using any max-flow algorithm. We refer to (Ahuja *et al.*, 1993) for an overview of efficient max-flow algorithms. Note also that the max-flow computation can be stopped prematurely whenever a flow of size k has been found. Therefore, an efficient alternative is to try k iterations of Edmonds-Karp algorithm, in which a single augmenting path can be determined in $\mathcal{O}(|A|) = \mathcal{O}(|E| + |V|)$ time using breadth-first search (BFS). Since our graphs are connected ($\mathcal{O}(|V|) \leq \mathcal{O}(|E|)$), the worst-case runtime complexity for a single test $\kappa_G(i,j) \geq k$ is $\mathcal{O}(k \cdot |E|)$. Finally, the straightforward verification must check a quadratic number of pairs and the test value k grows linearly with the current set P' so that the overall worst-case runtime is $\mathcal{O}(|P'|^3 \cdot |E|)$.

We can reduce the worst-case runtime by one order of magnitude using the following theorem (slightly shortened and reformulated with the symbols we use here):

Theorem (Kleitman, 1969)**.** In order to verify the existence of k vertex-disjoint paths between each pair of vertices in $G = (V, E)$ is suffices to choose any vertex $r \in V$ and to verify

(i) the existence of k vertex-disjoint paths between r and vertices of $V \setminus \{r\}$;

(ii) the existence of $k-1$ vertex-disjoint paths between each pair of vertices in $G[V \setminus \{r\}]$.

To verify the latter condition, the criterion can be used recursively.

The direct consequence for our verification procedure is the following: Assume that P' is an s-bundle. Then $P' \cup \{v\}$ is an s-bundle if and only if there exist $|P'| - s$ vertex-disjoint paths between v and vertices of P' in $G[P' \cup \{v\}]$. Hence, the runtime complexity of the incremental verification procedure reduces to $\mathcal{O}(|P'|^2 \cdot |E(G[P'])|)$ (recall that one factor $|P'|$ results from checking the paths to each vertex in P', the other factor $|P'|$ results from the repeated calls to the BFS augmenting path search, which requires no more than $\mathcal{O}(|E(G[P'])|)$ steps).

Now, we briefly describe the data structures needed within the RDS recursion: Let $\textsc{star}[v]$ be the star of vertex $v \in P'$ in $G[P']$. There is a one-to-one correspondence between vertices and edges of $G[P']$ with the vertices and arcs of the corresponding network $\mathcal{N}[P'^+ \cup P'^-]$ induced by $P'^+ = \{u^+ : u \in P'\}$ and $P'^- = \{u^- : u \in P'\}$. Hence, $\textsc{star}[v]$ for all $v \in P'$ delivers an implicit representation of the network $\mathcal{N}[P'^+ \cup P'^-]$, in which the BFS-based augmenting path computation can be performed. For checking if $P' \cup \{v\}$ is an s-bundle for some candidate $v \in C$, we temporarily add the edges $\{v, u\}$ for adjacent vertices v and $u \in P'$ to the stars. Such a modification can be done and undone in $\mathcal{O}(|P'|)$ time before and after the actual path computations. Keeping $\textsc{star}[v]$ updated over the recursive calls of FindMax guarantees that there is no additional effort needed to set up the network. Hence, the incremental verification procedure runs in $\mathcal{O}(|P'|^2 \cdot |E(G[P'])|)$ time.

Max Π problem	Trukhanov *et al.* (2013) Update step	Verification	New Update step	Verification
s-plex	?	$\mathcal{O}(\lvert\text{SAT}\rvert)$	$\mathcal{O}(\lvert P'\rvert + \lvert C\rvert)$	$\mathcal{O}(\lvert\text{SAT}\rvert)$
s-defective clique	?	$\mathcal{O}(\lvert P'\rvert^2)$	$\mathcal{O}(\lvert C\rvert)$	$\mathcal{O}(1)$
s-bundle	—	—	$\mathcal{O}(\lvert P'\rvert + \lvert C\rvert)$	$\mathcal{O}(\lvert P'\rvert^2 \cdot \lvert E(G[P'])\rvert)$

Table 1: Runtime complexity of the incremental verification procedures
Note: candidate set C; current set P'; saturated vertices SAT

The update of the STAR$[v]$ data structure works as follows: When P' is extended by vertex v_i in Step 10, all edges $\{v_i, v\}$ for $v \in P'$ are added to the stars of v_i and v, respectively. This takes no more than $\mathcal{O}(|P'|)$ time.

Acceleration of the average case. For many candidate vertices $v \in C$, showing that $P' \cup \{v\}$ is no s-bundle can be checked with a simple pre-test. We suggest its use in order to accelerate the incremental verification procedure. Recall that any s-bundle is also an s-plex. The s-plex verification procedures can be efficiently implemented as discussed above using the NNCNT$[v]$ data structure. With it, providing the auxiliary information in Step 10 become slightly more complex as the worst-case effort increases from $\mathcal{O}(|P'|)$ to $\mathcal{O}(|P'| + |C|)$. Our computational tests have, however, confirmed that the actual verification becomes much faster.

2.4. Overview of the computational complexity

Table 1 provides an overview of the worst-case runtime complexity for both key components, the update step (Step 10) and the Π verification (Step 11 of Procedure FindMax). The comparison with the work of Trukhanov *et al.* (2013) is not possible for s-plex due to their incomplete description of the verification procedure. We suspect however that they implemented the verification procedure in the way it is presented here.

3. Computational results

All computations were performed on a single thread of a standard PC with an Intel(R) Core(TM) i7-2600 processor at 3.4 GHz with 16 GB of main memory. Algorithms were coded in C++ and compiled in release mode with MS Visual Studio 2010(TM). The time limit was set to 600 seconds.

For our computational study we have chosen an extended set from four families of benchmark instances. The first set stems from the 2nd DIMACS challenge (`http://dimacs.rutgers.edu/Challenges/`) and comprises 66 clique instances. Trukhanov *et al.* (2013) used a proper subset of 26 clique instances. The second set is taken from the 10th DIMACS challenge (same URL) with 29 graph partitioning and clustering instances, from which 23 were used by Trukhanov *et al.* The third set are the 14 instances from the Stanford Network Analysis Project (SNAP, `http://snap.stanford.edu/`) considered by Trukhanov *et al.* The fourth set consists of 136 graph coloring instances taken from `https://sites.google.com/site/graphcoloring/home` not analyzed by Trukhanov *et al.* In order to reduce the graph sizes, the so-called *peeling procedure* (Abello *et al.*, 1999) is applied to all instances. It recursively removes vertices of degree less that $\omega(G) - s$, where $\omega(G)$ is a lower bound on the clique number. The removal of these vertices does not affect maximum s-plex, $(s-1)$-defective cliques, and s-bundles.

In pre-tests we found that the vertex ordering has only a minor impact on the computation times for the benchmark instances. Therefore, we run the RDS algorithm with the default vertex ordering as given by the input file.

Table 2 presents for the maximum s-plex, s-defective clique, and s-bundle problems and each of the four benchmark sets how many of the n instances can be solved to proven optimality within the time limit. For each of the problems, we consider four different values of s. It can be seen that for all problems the difficulty

		s-Plex for $s =$				s-Defective for $s =$				s-Bundle for $s =$			
Group	n	2	3	4	5	1	2	3	4	2	3	4	5
2nd DIMACS	66	21	14	11	11	26	22	17	15	18	13	9	8
10th DIMACS	29	29	29	27	26	29	29	29	28	29	26	23	18
SNAP	14	12	5	4	2	12	10	5	1	5	1	1	0
Coloring	136	114	101	88	66	114	112	97	94	109	91	60	42
Total	245	176	149	130	105	181	173	148	138	161	131	93	68
$s = 2, \ldots, 5$ or $s = 1, \ldots, 4$	980	560				640				453			

Table 2: Number of instances solved to proven optimality within 600s

increases with s. Moreover, Table 2 indicates that the maximum s-bundle problem is the hardest of the three problems while the maximum s-defective clique problem appears to be the easiest. This is in line with the complexity of the verification procedures as given in Table 1.

A comparison between the original maximum cardinality s-defective clique algorithm of Trukhanov *et al.* (2013) and our new algorithm is shown in Table 3. It includes only those instances that were considered by Trukhanov *et al.* The first three blocks compare the number of optimally solved instances per benchmark set for three algorithms: Data in the first block is taken from the paper (Trukhanov *et al.*, 2013). Since they allowed much longer computation times of up to 3 hours (10800s), we report both the numbers for our time limit of 600s and the additional numbers for their time limit of 10800s. The second block is for our implementation of their algorithm (using the s-plex pre-test together with the quadratic verification procedure) and the third block is for our new algorithm.

Overall, Trukhanov *et al.* solved 177 instances while our re-implementation of their algorithm solved 180 instances. These number seem comparable. Our new algorithm with the faster incremental verification procedure computed 191 proven optimal solutions.

Note that comprehensive tables with detailed characteristics and results (size after peeling, runtime, optimum or best bound) for each instance are given in the Appendix.

The forth block of Table 3 compares the computation times for the two algorithms we implemented. It is not reasonable to directly compare computation times between different machines, implementations, and compilers. Hence, we present no runtime comparison between results from the paper (Trukhanov *et al.*, 2013) and ours. The factor shown in the last block of Table 3 is the runtime of the re-implementation of the algorithm by Trukhanov *et al.* divided by the runtime of the new algorithm. The geometric mean is taken only over those instances for which both algorithms terminated before the time limit. In order to gain higher precision, we have performed 1000 calls to the RDS whenever computation times were below 0.1s. In summary, the new algorithm is by factor 3.6 faster than our re-implementation of the algorithm by Trukhanov *et al.*

4. Conclusion

This note builds on the work of Trukhanov *et al.* (2013) who apply the Russian Doll Search (RDS) principle for identifying maximum cardinality and maximum weight s-plex and s-defective cliques. These are hereditary structures of increasing interest in social network analysis and beyond. A key component to make RDS algorithms effective is a fast verification procedure needed to distinguish between feasible and infeasible structures. We have presented an alternative incremental verification procedure for s-defective cliques, which reduces the worst-case run time from quadratic to linear. Computational results on benchmark instances from the literature indicate that overall computation times of the RDS reduce by a factor of 3.6 (on average). Furthermore, we have designed an incremental verification procedure for s-bundle in order to present a first exact algorithm for this problem. It utilizes that the s-bundle property can be locally and

Group	n	Trukhanov et al. 2013				Trukhanov Our code‡				New*				Factor time ‡/*			
		s-Defective for $s =$															
		1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
2nd DIMACS	26	19^{+1}	15^{+1}	13^{+3}	11^{+2}	20	18	15	13	22	20	16	14	3.4	4.1	4.7	5.4
10th DIMACS	23	23^{+0}	23^{+0}	20^{+0}	18^{+1}	23	23	23	21	23	23	23	22	3.7	3.4	3.9	3.8
SNAP	14	12^{+2}	6^{+1}	3^{+0}	3^{+0}	11	8	5	0	12	10	5	1	2.3	1.7	2.0	—
Total	63	54^{+3}	44^{+2}	36^{+3}	32^{+3}	54	49	43	34	57	53	44	37	3.3	3.3	3.9	4.3
$s = 1, \ldots, 4$	252	166^{+11}				180				191				3.6			

Table 3: Comparison of three RDS for s-defective clique: Number of instances solved to proven optimality and runtime factor

efficiently tested using an auxiliary network in which a sufficient number of vertex-disjoint paths have to be found. Fortunately, augmenting path algorithms well-known in the context of max-flow computations can be used. Moreover, the number of augmenting path computations can be reduced by one order of magnitude if a straightforward verification procedure is replaced by an incremental verification procedure. Computational results show that maximum cardinality s-bundle problems can be solved to optimality for many benchmark instances, even if verification here requires more effort compared to s-plex and s-defective clique.

References

Abello, J., Pardalos, P., and Resende, M. G. C. (1999). On maximum clique problems in very large graphs. In J. M. Abello and J. S. Vitter, editors, *External Memory Algorithms and Visualization. DIMACS Series on Discrete Mathematics and Theoretical Computer Science,*, pages 119–130. American Mathematical Society, Boston, MA, USA.

Ahuja, R., Magnanti, T., and Orlin, J. (1993). *Network Flows: Theory, Algorithms, and Applications.* Prentice Hall, Englewood Cliffs, New Jersey.

Carraghan, R. and Pardalos, P. M. (1990). An exact algorithm for the maximum clique problem. *Operations Research Letters*, **9**(6), 375–382.

Fortunato, S. (2010). Community detection in graphs. *Physics Reports*, **486**(3–5), 75–174.

Kammer, F. and Täubig, H. (2004). Connectivity. In U. Brandes and T. Erlebach, editors, *Network Analysis*, volume 3418 of *Lecture Notes in Computer Science*, pages 143–177. Springer.

Kleitman, D. (1969). Methods for investigating connectivity of large graphs. *IEEE Transactions on Circuit Theory*, **16**(2), 232–233.

Menger, K. (1927). Zur allgemeinen Kurventheorie. *Fund. Math.*, **10**, 96–115.

Östergård, P. R. J. (2002). A fast algorithm for the maximum clique problem. *Discrete Applied Mathematics*, **120**(1-3), 197–207. Special Issue devoted to the 6th Twente Workshop on Graphs and Combinatorial Optimization.

Pattillo, J., Youssef, N., and Butenko, S. (2013). On clique relaxation models in network analysis. *European Journal of Operational Research*, **226**(1), 9–18.

Trukhanov, S., Balasubramaniam, C., Balasundaram, B., and Butenko, S. (2013). Algorithms for detecting optimal hereditary structures in graphs, with application to clique relaxations. *Computational Optimization and Applications*, **56**(1), 113–130.

Verfaillie, G., Lemaître, M., and Schiex, T. (1996). Russian doll search for solving constraint optimization problems. In *Proceedings of the thirteenth national conference on Artificial intelligence*, volume 1, pages 181–187. AAAI Press.

Yannakakis, M. (1978). Node-and edge-deletion NP-complete problems. In *STOC '78: Proceedings of the 10th Annual ACM Symposium on Theory of Computing*, pages 253–264, New York, NY. ACM Press.

Appendix

This appendix lists characteristics and detailed computational results for each instance. The following information is given:

Graph	problem instance
$\|V\|$	number of vertices
$\|E\|$	number of edges
$\|\rho(G)\|$	density of the graph $G = (V, E)$ in %
$\|\omega(G)\|$	clique number (or lower bound) of $G = (V, E)$ used for the peeling procedure
s	parameter s for s-plex, s-defective clique, and s-bundle
$\|V^{red}\|$	number of remaining vertices after application of the peeling procedure
opt	cardinality of a maximum relaxed clique, $\geq$ indicates that only a lower bound was provided within the time limit of 600s,
time	computation time in seconds, times smaller than 0.01s are rounded up to 0.01s, `OoM` indicates that the maximum s-bundle algorithm algorithm ran out of memory

All benchmark instances and in particular the reduced instances resulting from the application of the peeling procedure are available on our website `http://logistik.bwl.uni-mainz.de/Dateien/RelaxedClique.zip`.

Table 4: Detailed results for s-Plex and Instances from the 2nd DIMACS challenge

					2-plex			3-plex			4-plex			5-plex		
Graph	$\|V\|$	$\|E\|$	$\rho(G)$	$\omega(G)$	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time
brock200_1.clq	200	14834	74.543	21	200	≥23	600.00	200	≥24	600.00	200	≥26	600.00	200	≥25	600.00
brock200_2.clq	200	9876	49.628	12	200	13	9.58	200	≥15	600.00	200	≥16	600.00	200	≥16	600.00
brock200_3.clq	200	12048	60.543	15	200	17	184.82	200	≥18	600.00	200	≥18	600.00	200	≥20	600.00
brock200_4.clq	200	13089	65.774	17	200	≥19	600.00	200	≥20	600.00	200	≥21	600.00	200	≥21	600.00
brock400_1.clq	400	59723	74.841	27	400	≥23	600.00	400	≥25	600.00	400	≥26	600.00	400	≥26	600.00
brock400_2.clq	400	59786	74.920	29	400	≥22	600.00	400	≥23	600.00	400	≥23	600.00	400	≥26	600.00
brock400_3.clq	400	59681	74.788	31	400	≥23	600.00	400	≥24	600.00	400	≥26	600.00	400	≥28	600.00
brock400_4.clq	400	59765	74.894	33	400	≥23	600.00	400	≥22	600.00	400	≥24	600.00	400	≥27	600.00
brock800_1.clq	800	207505	64.927	23	800	≥19	600.00	800	≥19	600.00	800	≥21	600.00	800	≥22	600.00
brock800_2.clq	800	208166	65.133	24	800	≥19	600.00	800	≥20	600.00	800	≥21	600.00	800	≥22	600.00
brock800_3.clq	800	207333	64.873	25	800	≥19	600.00	800	≥20	600.00	800	≥21	600.00	800	≥21	600.00
brock800_4.clq	800	207643	64.970	26	800	≥19	600.00	800	≥20	600.00	800	≥21	600.00	800	≥20	600.00
c-fat200-1.clq	200	1534	7.709	12	200	12	0.01	200	12	0.01	200	12	0.01	200	14	0.02
c-fat200-2.clq	200	3235	16.256	24	200	24	0.01	200	24	0.01	200	24	0.02	200	24	0.75
c-fat200-5.clq	200	8473	42.578	58	200	58	0.01	200	58	0.02	200	58	0.05	200	58	0.94
c-fat500-1.clq	500	4459	3.574	14	500	14	0.01	500	14	0.02	500	14	0.03	500	15	0.06
c-fat500-10.clq	500	46627	37.376	126	500	126	0.03	500	126	0.05	500	126	0.30	500	126	3.88
c-fat500-2.clq	500	9139	7.326	26	500	26	0.01	500	26	0.01	500	26	0.05	500	26	1.37
c-fat500-5.clq	500	23191	18.590	64	500	64	0.01	500	64	0.02	500	64	0.09	500	64	1.62
hamming10-2.clq	1024	518656	99.023	512	1024	512	26.63	1024	≥89	600.00	1024	≥44	600.00	1024	≥55	600.00
hamming10-4.clq	1024	434176	82.894	40	1024	≥22	600.00	1024	≥16	600.00	1024	≥18	600.00	1024	≥16	600.00
hamming6-2.clq	64	1824	90.476	32	64	32	0.02	64	32	0.23	64	40	298.34	64	48	32.95
hamming6-4.clq	64	704	34.921	4	64	6	0.01	64	8	0.02	64	10	0.13	64	12	1.65
hamming8-2.clq	256	31616	96.863	128	256	128	0.11	256	≥89	600.00	256	≥44	600.00	256	≥55	600.00
hamming8-4.clq	256	20864	63.922	16	256	16	9.03	256	≥16	600.00	256	≥18	600.00	256	≥16	600.00
johnson16-2-4.clq	120	5460	76.471	8	120	≥10	600.00	120	≥15	600.00	120	≥18	600.00	120	≥20	600.00
johnson32-2-4.clq	496	107880	87.879	16	496	≥21	600.00	496	≥24	600.00	496	≥25	600.00	496	≥26	600.00
johnson8-2-4.clq	28	210	55.556	4	28	5	0.01	28	8	0.01	28	9	0.03	28	12	0.05
johnson8-4-4.clq	70	1855	76.812	14	70	14	0.02	70	18	5.91	70	≥21	600.00	70	≥23	600.00
keller4.clq	171	9435	64.912	11	171	15	110.68	171	≥21	600.00	171	≥16	600.00	171	≥18	600.00
keller5.clq	776	225990	75.155	27	776	≥15	600.00	776	≥22	600.00	776	≥16	600.00	776	≥18	600.00
keller6.clq	3361	4619898	81.819	59	3361	≥15	600.00	3361	≥22	600.00	3361	≥16	600.00	3361	≥18	600.00
MANN_a27.clq	378	70551	99.015	126	378	≥235	600.00	378	≥351	600.00	378	≥351	600.00	378	≥351	600.00
MANN_a45.clq	1035	533115	99.630	345	1035	≥661	600.00	1035	≥990	600.00	1035	≥990	600.00	1035	≥990	600.00
MANN_a81.clq	3321	5506380	99.883	1100	3321	≥1487	600.00	3321	≥1673	600.00	3321	≥1659	600.00	3321	≥1646	600.00
MANN_a9.clq	45	918	92.727	16	45	26	0.03	45	36	0.37	45	36	36.19	45	45	0.01
p_hat1000-1.clq	1000	122253	24.475	10	1000	≥13	600.00	1000	≥13	600.00	1000	≥14	600.00	1000	≥15	600.00
p_hat1000-2.clq	1000	244799	49.009	46	1000	≥30	600.00	1000	≥27	600.00	1000	≥26	600.00	1000	≥26	600.00
p_hat1000-3.clq	1000	371746	74.424	68	1000	≥33	600.00	1000	≥30	600.00	1000	≥33	600.00	1000	≥35	600.00
p_hat1500-1.clq	1500	284923	25.343	12	1500	≥13	600.00	1500	≥14	600.00	1500	≥13	600.00	1500	≥14	600.00
p_hat1500-2.clq	1500	568960	50.608	65	1500	≥27	600.00	1500	≥29	600.00	1500	≥31	600.00	1500	≥28	600.00
p_hat1500-3.clq	1500	847244	75.361	94	1500	≥34	600.00	1500	≥33	600.00	1500	≥34	600.00	1500	≥33	600.00

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Table 4 – Continued from previous page

Graph	$\|V\|$	$\|E\|$	$\rho(G)$	$\omega(G)$	2-plex			3-plex			4-plex			5-plex		
					$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time
p_hat300-1.clq	300	10933	24.377	8	300	10	0.75	300	12	75.19	300	≥13	600.00	300	≥14	600.00
p_hat300-2.clq	300	21928	48.892	25	300	≥29	600.00	300	≥27	600.00	300	≥27	600.00	300	≥28	600.00
p_hat300-3.clq	300	33390	74.448	36	300	≥31	600.00	300	≥32	600.00	300	≥31	600.00	300	≥31	600.00
p_hat500-1.clq	500	31569	25.306	9	500	12	13.45	500	≥13	600.00	500	≥14	600.00	500	≥14	600.00
p_hat500-2.clq	500	62946	50.458	36	500	≥34	600.00	500	≥31	600.00	500	≥29	600.00	500	≥30	600.00
p_hat500-3.clq	500	93800	75.190	50	500	≥35	600.00	500	≥35	600.00	500	≥33	600.00	500	≥35	600.00
p_hat700-1.clq	700	60999	24.933	11	700	13	50.84	700	≥13	600.00	700	≥13	600.00	700	≥13	600.00
p_hat700-2.clq	700	121728	49.756	44	700	≥31	600.00	700	≥30	600.00	700	≥29	600.00	700	≥25	600.00
p_hat700-3.clq	700	183010	74.805	62	700	≥32	600.00	700	≥29	600.00	700	≥29	600.00	700	≥30	600.00
san1000.clq	1000	250500	50.150	15	1000	≥16	600.00	1000	≥24	600.00	1000	≥30	600.00	1000	≥39	600.00
san200_0.7_1.clq	200	13930	70.000	30	200	≥29	600.00	200	≥41	600.00	200	≥52	600.00	200	≥73	600.00
san200_0.7_2.clq	200	13930	70.000	18	200	≥24	600.00	200	≥34	600.00	200	≥46	600.00	200	≥56	600.00
san200_0.9_1.clq	200	17910	90.000	70	200	≥67	600.00	200	125	197.07	200	≥38	600.00	200	≥40	600.00
san200_0.9_2.clq	200	17910	90.000	60	200	≥42	600.00	200	≥47	600.00	200	≥43	600.00	200	≥46	600.00
san200_0.9_3.clq	200	17910	90.000	44	200	≥42	600.00	200	≥35	600.00	200	≥38	600.00	200	≥43	600.00
san400_0.5_1.clq	400	39900	50.000	13	400	≥14	600.00	400	≥20	600.00	400	≥26	600.00	400	≥31	600.00
san400_0.7_1.clq	400	55860	70.000	40	400	≥34	600.00	400	≥48	600.00	400	≥70	600.00	400	≥90	600.00
san400_0.7_2.clq	400	55860	70.000	30	400	≥28	600.00	400	≥41	600.00	400	≥51	600.00	400	≥56	600.00
san400_0.7_3.clq	400	55860	70.000	22	400	≥23	600.00	400	≥33	600.00	400	≥44	600.00	400	≥54	600.00
san400_0.9_1.clq	400	71820	90.000	100	400	≥64	600.00	400	≥47	600.00	400	≥36	600.00	400	≥41	600.00
sanr200_0.7.clq	200	13868	69.688	18	200	≥20	600.00	200	≥21	600.00	200	≥22	600.00	200	≥24	600.00
sanr200_0.9.clq	200	17863	89.764	42	200	≥33	600.00	200	≥37	600.00	200	≥40	600.00	200	≥43	600.00
sanr400_0.5.clq	400	39984	50.105	13	400	≥15	600.00	400	≥16	600.00	400	≥18	600.00	400	≥17	600.00
sanr400_0.7.clq	400	55869	70.011	21	400	≥20	600.00	400	≥21	600.00	400	≥23	600.00	400	≥24	600.00

Table 5: Detailed results for s-Plex and Instances from the 10th DIMACS challenge

Graph	$\|V\|$	$\|E\|$	$\rho(G)$	$\omega(G)$	2-plex $\|V^{red}\|$	2-plex opt	2-plex time	3-plex $\|V^{red}\|$	3-plex opt	3-plex time	4-plex $\|V^{red}\|$	4-plex opt	4-plex time	5-plex $\|V^{red}\|$	5-plex opt	5-plex time
adjnoun.graph	112	425	6.837	5	89	6	0.02	102	8	0.02	112	8	0.14	112	10	0.62
as-22july06.graph	22963	48436	0.018	17	168	19	0.02	182	21	0.20	204	22	8.05	232	24	279.16
astro-ph.graph	16706	121251	0.087	57	113	57	0.01	113	57	0.01	165	57	0.56	165	57	18.60
caidaRouterLevel.graph	192244	609066	0.003	17	4021	20	2.17	4704	23	263.60	5417	≥10	600.00	6447	≥12	600.00
celegans_metabolic.graph	453	2025	1.978	9	92	10	0.01	138	11	0.05	240	13	1.70	313	14	134.50
celegansneural.graph	297	2148	4.887	8	251	10	0.02	265	11	0.20	274	12	5.01	278	13	83.84
chesapeake.graph	39	170	22.942	5	39	7	0.01	39	8	0.01	39	9	0.02	39	11	0.05
cnr-2000.graph	325557	2738969	0.005	84	86	85	0.02	89	86	0.01	170	86	0.02	286	≥80	600.00
coAuthorsCiteseer.graph	227320	814134	0.003	87	87	87	0.02	87	87	0.02	87	87	0.02	87	87	0.02
coAuthorsDBLP.graph	299067	977676	0.002	115	115	115	0.02	115	115	0.02	115	115	0.02	115	115	0.02
cond-mat.graph	16726	47594	0.034	18	18	18	0.02	53	18	0.02	98	19	0.01	164	20	0.05
cond-mat-2003.graph	31163	120029	0.025	25	27	25	0.01	50	26	0.01	77	27	0.05	77	27	0.09
cond-mat-2005.graph	40421	175691	0.022	30	30	30	0.01	30	30	0.01	57	30	0.02	83	30	0.01
dolphins.graph	62	159	8.408	5	45	6	0.01	53	7	0.01	62	7	0.01	62	9	0.02
email.graph	1133	5451	0.850	12	121	12	0.02	238	12	0.17	349	12	4.10	434	13	35.46
football.graph	115	613	9.352	9	115	10	0.02	115	11	0.01	115	12	0.02	115	12	0.01
hep-th.graph	8361	15751	0.045	24	24	24	0.02	24	24	0.01	24	24	0.01	24	24	0.01
jazz.graph	198	2742	14.059	30	30	30	0.01	30	30	0.01	30	30	0.01	30	30	0.01
karate.graph	34	78	13.904	5	22	6	0.01	33	6	0.01	34	8	0.01	34	9	0.01
lesmis.graph	77	254	8.681	10	20	10	0.01	31	12	0.01	38	12	0.02	38	12	0.02
memplus.graph	17758	54196	0.034	97	97	97	0.02	97	97	0.02	97	97	0.02	97	97	0.02
netscience.graph	1589	2742	0.217	20	20	20	0.01	20	20	0.01	20	20	0.01	20	20	0.01
PGPgiantcompo.graph	10680	24316	0.043	25	126	29	0.02	145	31	0.03	171	33	0.11	172	35	0.47
polblogs.graph	1490	16715	1.507	20	459	23	1.06	489	27	17.27	517	≥29	600.00	541	≥20	600.00
polbooks.graph	105	441	8.077	6	98	7	0.01	103	9	0.01	105	10	0.03	105	11	0.11
power.graph	4941	6594	0.054	6	36	6	0.01	231	6	0.01	3353	8	0.11	4941	9	0.17
rgg_n_2_17_s0.graph	131072	728474	0.008	15	125	16	0.01	650	17	0.01	2002	18	0.03	6428	18	0.44
rgg_n_2_19_s0.graph	524288	3269220	0.002	18	55	19	0.02	211	19	0.01	534	20	0.01	1995	21	0.05
rgg_n_2_20_s0.graph	1048576	6890866	0.001	17	462	18	0.01	1966	19	0.03	6339	20	0.38	19576	20	5.18

Table 6: Detailed results for s-Plex and Instances from the SNAP

					2-plex			3-plex			4-plex			5-plex		
Graph	$\|V\|$	$\|E\|$	$\rho(G)$	$\omega(G)$	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time
Cit-HepPh.txt	34546	420877	0.071	19	11284	24	19.02	12471	≥25	600.00	13697	≥21	600.00	14992	≥14	600.00
Cit-HepTh.txt	27769	352285	0.091	23	7278	28	224.01	7743	≥31	600.00	8167	≥33	600.00	8595	≥36	600.00
Email-EuAll.txt	265009	364481	0.001	16	1852	19	1.83	2026	22	90.54	2227	≥11	600.00	2470	≥5	600.00
p2p-Gnutella04.txt	10876	39994	0.068	4	8379	5	0.94	10876	7	3.28	10876	9	12.83	10876	10	58.61
p2p-Gnutella24.txt	26518	65369	0.019	4	15519	5	2.63	26518	6	23.77	26518	8	540.17	26518	≥7	600.00
p2p-Gnutella25.txt	22687	54705	0.021	4	13353	5	1.91	22687	6	7.63	22687	8	9.06	22687	10	15.32
Slashdot0811.txt	77360	469180	0.016	26	5418	31	45.99	5727	≥8	600.00	6142	≥7	600.00	6571	≥8	600.00
Slashdot0902.txt	82168	504230	0.015	27	5417	32	27.33	5734	≥8	600.00	6093	≥9	600.00	6539	≥10	600.00
soc-Epinions1.txt	75879	405740	0.014	23	5243	28	277.14	5456	≥27	600.00	5719	≥21	600.00	6010	≥22	600.00
web-BerkStan.txt	685230	6649470	0.003	201	392	202	0.19	392	202	2.18	392	202	112.01	392	≥162	600.00
web-Google.txt	875713	4322051	0.001	44	218	≥44	600.00	222	≥45	600.00	223	≥46	600.00	223	≥46	600.00
web-NotreDame.txt	325729	1090108	0.002	155	1367	155	4.88	1367	≥152	600.00	1367	≥150	600.00	1367	≥150	600.00
web-Stanford.txt	281903	1992636	0.005	61	1389	≥59	600.00	1439	≥59	600.00	1499	≥5	600.00	1595	≥5	600.00
Wiki-Vote.txt	7115	100762	0.398	17	2382	21	11.61	2452	≥23	600.00	2520	≥13	600.00	2604	≥6	600.00

Table 7: Detailed results for s-Plex and Instances from the coloring benchmark set

					2-plex			3-plex			4-plex			5-plex		
Graph	$\|V\|$	$\|E\|$	$\rho(G)$	$\omega(G)$	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time
1-FullIns_3.col	30	100	22.989	3	30	5	0.01	30	7	0.01	30	8	0.01	30	8	0.03
1-FullIns_4.col	93	593	13.862	3	93	6	0.01	93	7	0.01	93	9	0.06	93	10	0.30
1-FullIns_5.col	282	3247	8.195	3	282	6	0.02	282	8	0.56	282	10	7.16	282	11	94.51
1-Insertions_4.col	67	232	10.493	2	67	4	0.01	67	5	0.01	67	6	0.20	67	8	1.95
1-Insertions_5.col	202	1227	6.044	2	202	4	0.01	202	6	0.05	202	8	0.30	202	9	3.01
1-Insertions_6.col	607	6337	3.446	2	607	4	0.13	607	6	2.23	607	8	57.08	607	$\geq$6	600.00
2-FullIns_3.col	52	201	15.158	4	52	5	0.01	52	7	0.01	52	8	0.01	52	9	0.01
2-FullIns_4.col	212	1621	7.248	4	212	6	0.01	212	8	0.08	212	10	0.70	212	11	4.38
2-FullIns_5.col	852	12201	3.366	4	852	7	0.31	852	8	8.05	852	10	280.12	852	$\geq$7	600.00
2-Insertions_3.col	37	72	10.811	2	37	4	0.02	37	4	0.01	37	6	0.02	37	7	0.06
2-Insertions_4.col	149	541	4.907	2	149	4	0.01	149	5	0.02	149	6	8.94	149	8	210.03
2-Insertions_5.col	597	3936	2.212	2	597	4	0.05	597	6	0.41	597	8	10.14	597	9	240.44
3-FullIns_3.col	80	346	10.949	5	80	6	0.01	80	7	0.01	80	8	0.01	80	10	0.02
3-FullIns_4.col	405	3524	4.308	5	405	7	0.03	405	9	0.36	405	10	4.88	405	11	52.06
3-FullIns_5.col	2030	33751	1.639	5	2030	8	1.70	2030	10	95.18	2030	$\geq$6	600.00	2030	$\geq$7	600.00
3-Insertions_3.col	56	110	7.143	2	56	4	0.01	56	4	0.01	56	6	0.08	56	7	0.75
3-Insertions_4.col	281	1046	2.659	2	281	4	0.02	281	5	0.01	281	6	184.52	281	$\geq$8	600.00
3-Insertions_5.col	1406	9695	0.982	2	1406	4	0.14	1406	6	3.87	1406	8	194.21	1406	$\geq$6	600.00
4-FullIns_3.col	114	541	8.399	6	114	7	0.02	114	8	0.02	114	9	0.03	114	10	0.06
4-FullIns_4.col	690	6650	2.798	6	690	8	0.06	690	10	1.28	690	12	24.88	690	12	419.47
4-FullIns_5.col	4146	77305	0.900	6	4146	9	8.02	4146	$\geq$6	600.00	4146	$\geq$6	600.00	4146	$\geq$7	600.00
4-Insertions_3.col	79	156	5.063	2	79	4	0.01	79	4	0.02	79	6	0.42	79	7	5.54
4-Insertions_4.col	475	1795	1.594	2	475	4	0.02	475	5	0.03	475	$\geq$6	600.00	475	$\geq$8	600.00
5-FullIns_3.col	154	792	6.723	7	136	8	0.01	154	9	0.01	154	10	0.06	154	11	0.22
5-FullIns_4.col	1085	11395	1.938	7	1085	9	0.19	1085	11	4.51	1085	13	102.09	1085	$\geq$10	600.00
abb313GPIA.col	1557	53356	4.405	8	1552	$\geq$14	600.00	1552	$\geq$17	600.00	1555	$\geq$21	600.00	1555	$\geq$23	600.00
anna.col	138	493	5.215	11	19	11	0.01	19	11	0.01	24	12	0.01	44	13	0.06
ash331GPIA.col	662	4181	1.911	3	662	4	0.03	662	6	0.25	662	8	1.97	662	10	15.79
ash608GPIA.col	1216	7844	1.062	3	1216	4	0.05	1216	6	0.45	1216	8	2.81	1216	10	16.36
ash958GPIA.col	1916	12506	0.682	3	1916	4	0.08	1916	6	0.75	1916	8	5.62	1916	10	39.17
C2000.5.col	2000	999836	50.017	16	2000	$\geq$15	600.00	2000	$\geq$15	600.00	2000	$\geq$17	600.00	2000	$\geq$16	600.00
C4000.5.col	4000	4000268	50.016	18	4000	$\geq$15	600.00	4000	$\geq$15	600.00	4000	$\geq$16	600.00	4000	$\geq$17	600.00
david.col	87	406	10.853	11	22	11	0.01	33	11	0.01	36	13	0.01	44	14	0.03
DSJC1000.1.col	1000	49629	9.936	6	1000	7	3.56	1000	$\geq$8	600.00	1000	$\geq$8	600.00	1000	$\geq$9	600.00
DSJC1000.5.col	1000	249826	50.015	15	1000	$\geq$15	600.00	1000	$\geq$16	600.00	1000	$\geq$16	600.00	1000	$\geq$17	600.00
DSJC1000.9.col	1000	449449	89.980	68	1000	$\geq$33	600.00	1000	$\geq$36	600.00	1000	$\geq$39	600.00	1000	$\geq$42	600.00
DSJC125.1.col	125	736	9.497	4	125	5	0.01	125	7	0.03	125	8	0.30	125	9	2.73
DSJC125.5.col	125	3891	50.207	10	125	13	0.44	125	14	47.39	125	$\geq$16	600.00	125	$\geq$17	600.00
DSJC125.9.col	125	6961	89.819	34	125	$\geq$34	600.00	125	$\geq$36	600.00	125	$\geq$39	600.00	125	$\geq$43	600.00
DSJC250.1.col	250	3218	10.339	4	250	6	0.03	250	7	1.76	250	8	97.11	250	$\geq$9	600.00
DSJC250.5.col	250	15668	50.339	12	250	14	62.74	250	$\geq$15	600.00	250	$\geq$16	600.00	250	$\geq$18	600.00
DSJC250.9.col	250	27897	89.629	43	250	$\geq$35	600.00	250	$\geq$34	600.00	250	$\geq$36	600.00	250	$\geq$42	600.00

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Table 7 – Continued from previous page

Graph	$\|V\|$	$\|E\|$	$\rho(G)$	$\omega(G)$	2-plex $\|V^{red}\|$	2-plex opt	2-plex time	3-plex $\|V^{red}\|$	3-plex opt	3-plex time	4-plex $\|V^{red}\|$	4-plex opt	4-plex time	5-plex $\|V^{red}\|$	5-plex opt	5-plex time
DSJC500.1.col	500	12458	9.986	5	500	6	0.45	500	8	40.36	500	≥9	600.00	500	≥9	600.00
DSJC500.5.col	500	62624	50.200	13	500	≥15	600.00	500	≥16	600.00	500	≥16	600.00	500	≥17	600.00
DSJC500.9.col	500	112437	90.130	56	500	≥33	600.00	500	≥36	600.00	500	≥42	600.00	500	≥43	600.00
DSJR500.1.col	500	3555	2.850	11	201	14	0.01	328	15	0.01	423	15	0.03	441	16	0.05
DSJR500.1c.col	500	121275	97.214	83	500	≥56	600.00	500	≥70	600.00	500	≥82	600.00	500	≥100	600.00
DSJR500.5.col	500	58862	47.184	122	488	≥79	600.00	489	≥73	600.00	492	≥61	600.00	492	≥55	600.00
flat1000_50_0.col	1000	245000	49.049	15	1000	≥14	600.00	1000	≥15	600.00	1000	≥16	600.00	1000	≥18	600.00
flat1000_60_0.col	1000	245830	49.215	15	1000	≥15	600.00	1000	≥15	600.00	1000	≥16	600.00	1000	≥18	600.00
flat1000_76_0.col	1000	246708	49.391	15	1000	≥15	600.00	1000	≥15	600.00	1000	≥16	600.00	1000	≥17	600.00
flat300_20_0.col	300	21375	47.659	11	300	14	115.38	300	≥15	600.00	300	≥16	600.00	300	≥17	600.00
flat300_26_0.col	300	21633	48.234	11	300	14	103.80	300	≥15	600.00	300	≥16	600.00	300	≥17	600.00
flat300_28_0.col	300	21695	48.372	12	300	14	122.10	300	≥15	600.00	300	≥17	600.00	300	≥17	600.00
fpsol2.i.1.col	496	11654	9.493	65	85	66	0.01	86	66	0.44	91	66	20.08	120	67	46.30
fpsol2.i.2.col	451	8691	8.565	30	165	31	0.02	203	31	0.38	238	32	17.10	260	≥12	600.00
fpsol2.i.3.col	425	8688	9.643	30	164	31	0.01	203	31	0.39	238	32	17.19	260	≥13	600.00
games120.col	120	638	8.936	9	120	10	0.01	120	10	0.01	120	10	0.01	120	12	0.02
homer.col	561	1628	1.036	13	35	13	0.01	61	13	0.01	68	14	0.02	98	15	0.67
huck.col	74	301	11.144	11	20	11	0.01	32	11	0.02	42	11	0.02	45	13	0.03
inithx.i.1.col	864	18707	5.018	54	122	55	2.28	143	56	8.86	150	56	34.96	158	57	174.02
inithx.i.2.col	645	13979	6.731	31	226	31	0.13	278	32	2.86	338	33	103.41	396	≥12	600.00
inithx.i.3.col	621	13969	7.256	31	212	31	0.14	268	32	2.36	335	33	96.52	396	≥11	600.00
jean.col	80	254	8.038	10	20	10	0.01	31	12	0.01	38	12	0.01	38	12	0.01
latin_square_10.col	900	307350	75.973	90	900	≥90	600.00	900	≥90	600.00	900	≥90	600.00	900	≥90	600.00
le450_15a.col	450	8168	8.085	15	414	15	0.05	419	15	1.56	420	15	66.21	427	≥13	600.00
le450_15b.col	450	8169	8.086	15	417	15	0.05	421	15	2.95	427	15	160.87	429	≥13	600.00
le450_15c.col	450	16680	16.511	15	450	15	0.22	450	15	19.11	450	≥16	600.00	450	≥13	600.00
le450_15d.col	450	16750	16.580	15	450	15	0.27	450	15	25.10	450	≥15	600.00	450	≥12	600.00
le450_25a.col	450	8260	8.176	25	272	25	0.02	280	25	0.23	289	25	4.76	297	25	131.18
le450_25b.col	450	8263	8.179	25	304	25	0.02	308	25	0.39	314	25	9.98	320	25	345.70
le450_25c.col	450	17343	17.167	25	436	25	0.14	438	25	8.35	439	25	428.13	442	≥16	600.00
le450_25d.col	450	17425	17.248	25	438	25	0.09	440	25	4.17	441	25	202.82	442	≥17	600.00
le450_5a.col	450	5714	5.656	5	450	6	0.08	450	8	2.75	450	9	92.24	450	≥10	600.00
le450_5b.col	450	5734	5.676	5	450	6	0.08	450	8	3.39	450	9	97.77	450	≥10	600.00
le450_5c.col	450	9803	9.704	5	450	7	0.14	450	9	17.21	450	≥10	600.00	450	≥10	600.00
le450_5d.col	450	9757	9.658	5	450	7	0.17	450	9	14.96	450	≥10	600.00	450	≥9	600.00
miles1000.col	128	3216	39.567	42	51	43	0.01	61	44	0.02	62	45	0.05	81	46	1.39
miles1500.col	128	5198	63.952	73	84	73	0.19	85	73	8.60	86	75	26.13	88	76	83.82
miles250.col	128	387	4.761	8	27	9	0.01	41	10	0.01	83	11	0.01	102	12	0.01
miles500.col	128	1170	14.395	20	29	21	0.02	35	22	0.01	36	23	0.01	36	24	0.02
miles750.col	128	2113	25.997	31	39	33	0.01	41	33	0.02	43	35	0.01	43	36	0.02
mug100_1.col	100	166	3.354	3	100	4	0.01	100	5	0.01	100	6	1.05	100	7	19.75
mug100_25.col	100	166	3.354	3	100	4	0.02	100	5	0.01	100	6	1.06	100	7	19.75
mug88_1.col	88	146	3.814	3	88	4	0.02	88	5	0.01	88	6	0.61	88	7	9.59

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Graph	$\|V\|$	$\|E\|$	$\rho(G)$	$\omega(G)$	2-plex $\|V^{red}\|$	opt	time	3-plex $\|V^{red}\|$	opt	time	4-plex $\|V^{red}\|$	opt	time	5-plex $\|V^{red}\|$	opt	time
mug88_25.col	88	146	3.814	3	88	4	0.02	88	5	0.01	88	6	0.59	88	7	9.45
mulsol.i.1.col	197	3925	20.331	49	56	50	0.01	57	51	0.03	63	51	0.11	65	52	0.13
mulsol.i.2.col	188	3885	22.102	31	116	31	0.11	119	32	0.13	122	34	0.30	124	34	1.69
mulsol.i.3.col	184	3916	23.260	31	117	31	0.11	120	32	0.13	123	34	0.31	125	34	1.89
mulsol.i.4.col	185	3946	23.185	31	118	31	0.11	121	32	0.14	124	34	0.48	126	34	1.97
mulsol.i.5.col	186	3973	23.092	31	119	31	0.11	122	32	0.14	125	34	0.36	127	34	1.98
myciel3.col	11	20	36.364	2	11	4	0.01	11	5	0.02	11	6	0.01	11	8	0.01
myciel4.col	23	71	28.063	2	23	4	0.01	23	5	0.01	23	6	0.01	23	8	0.02
myciel5.col	47	236	21.832	2	47	4	0.01	47	6	0.02	47	8	0.03	47	9	0.30
myciel6.col	95	755	16.909	2	95	4	0.01	95	6	0.08	95	8	1.01	95	10	12.98
myciel7.col	191	2360	13.006	2	191	4	0.05	191	6	1.45	191	8	34.87	191	≥10	600.00
qg.order100.col	10000	990000	1.980	100	10000	≥100	600.00	10000	≥100	600.00	10000	≥100	600.00	10000	≥100	600.00
qg.order30.col	900	26100	6.452	30	900	30	1.59	900	30	361.80	900	≥30	600.00	900	≥30	600.00
qg.order40.col	1600	62400	4.878	40	1600	40	9.22	1600	≥40	600.00	1600	≥40	600.00	1600	≥40	600.00
qg.order60.col	3600	212400	3.279	60	3600	60	99.28	3600	≥60	600.00	3600	≥60	600.00	3600	≥60	600.00
queen10_10.col	100	1470	29.697	10	100	10	0.02	100	10	0.44	100	10	10.03	100	13	181.94
queen11_11.col	121	1980	27.273	11	121	11	0.03	121	11	0.76	121	11	21.89	121	13	541.50
queen12_12.col	144	2596	25.214	12	144	12	0.02	144	12	1.40	144	12	45.13	144	≥13	600.00
queen13_13.col	169	3328	23.443	13	169	13	0.05	169	13	2.40	169	13	87.42	169	≥13	600.00
queen14_14.col	196	4186	21.905	14	196	14	0.06	196	14	3.85	196	14	161.48	196	≥14	600.00
queen15_15.col	225	5180	20.556	15	225	15	0.08	225	15	6.05	225	15	292.91	225	≥15	600.00
queen16_16.col	256	6320	19.363	16	256	16	0.11	256	16	9.05	256	16	497.78	256	≥16	600.00
queen5_5.col	25	160	53.333	5	25	6	0.01	25	9	0.02	25	10	0.01	25	13	0.02
queen6_6.col	36	290	46.032	6	36	6	0.02	36	9	0.01	36	10	0.06	36	13	0.22
queen7_7.col	49	476	40.476	7	49	7	0.01	49	9	0.03	49	10	0.33	49	13	2.22
queen8_12.col	96	1368	30.000	12	96	12	0.02	96	12	0.27	96	12	6.22	96	13	124.91
queen8_8.col	64	728	36.111	8	64	8	0.02	64	9	0.08	64	10	1.25	64	13	12.95
queen9_9.col	81	1056	32.593	9	81	9	0.01	81	9	0.20	81	10	3.81	81	13	53.48
r1000.1.col	1000	14378	2.878	20	463	21	0.02	665	22	0.09	844	23	0.59	924	24	4.71
r1000.1c.col	1000	485090	97.115	91	1000	≥56	600.00	1000	≥64	600.00	1000	≥87	600.00	1000	≥93	600.00
r1000.5.col	1000	238267	47.701	234	984	≥153	600.00	984	≥76	600.00	985	≥74	600.00	985	≥73	600.00
r125.1.col	125	209	2.697	5	57	6	0.01	101	6	0.01	122	7	0.01	125	9	0.01
r125.1c.col	125	7501	96.787	46	125	≥47	600.00	125	≥62	600.00	125	≥70	600.00	125	≥88	600.00
r125.5.col	125	3838	49.523	36	119	36	0.20	119	38	3.62	120	39	12.39	122	40	156.08
r250.1.col	250	867	2.786	8	70	8	0.02	140	9	0.01	203	11	0.01	234	11	0.01
r250.1c.col	250	30227	97.115	63	250	≥49	600.00	250	≥65	600.00	250	≥78	600.00	250	≥94	600.00
r250.5.col	250	14849	47.708	65	237	65	4.73	237	67	243.74	238	≥66	600.00	245	≥58	600.00
school1.col	385	19095	25.832	14	361	≥28	600.00	363	≥34	600.00	363	≥35	600.00	363	≥39	600.00
school1_nsh.col	352	14612	23.653	14	331	28	243.19	332	37	52.18	332	≥41	600.00	333	≥35	600.00
wap01a.col	2368	110871	3.956	41	2107	41	12.89	2166	≥41	600.00	2281	≥42	600.00	2288	≥28	600.00
wap02a.col	2464	111742	3.682	40	2248	40	17.94	2362	≥41	600.00	2372	≥40	600.00	2377	≥31	600.00
wap03a.col	4730	286722	2.564	40	4701	≥41	600.00	4702	≥40	600.00	4702	≥39	600.00	4717	≥29	600.00
wap04a.col	5231	294902	2.156	40	5204	41	160.04	5205	≥40	600.00	5207	≥33	600.00	5223	≥29	600.00

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Table 7 – Continued from previous page

					2-plex			3-plex			4-plex			5-plex		
Graph	$\|V\|$	$\|E\|$	$\rho(G)$	$\omega(G)$	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time
wap05a.col	905	43081	10.532	50	675	50	0.48	679	50	23.70	685	≥40	600.00	693	≥36	600.00
wap06a.col	947	43571	9.727	40	807	40	3.88	834	40	221.41	846	≥41	600.00	865	≥35	600.00
wap07a.col	1809	103368	6.321	40	1701	41	15.97	1710	≥42	600.00	1719	≥40	600.00	1724	≥27	600.00
wap08a.col	1870	104176	5.961	40	1753	40	17.43	1763	≥40	600.00	1773	≥40	600.00	1779	≥36	600.00
will199GPIA.col	701	6772	2.760	6	700	8	0.05	700	10	0.72	701	12	7.85	701	14	66.47
zeroin.i.1.col	211	4100	18.506	49	73	49	0.16	79	50	2.29	79	51	19.02	91	52	268.12
zeroin.i.2.col	211	3541	15.983	30	106	30	0.02	131	32	0.34	136	32	4.07	137	33	50.36
zeroin.i.3.col	206	3540	16.765	30	106	30	0.02	131	32	0.34	136	32	4.06	137	33	51.25

Table 8: Detailed results for s-Defective clique and Instances from the 2nd DIMACS challenge

					1-defective clique			2-defective clique			3-defective clique			4-defective clique		
Graph	$\|V\|$	$\|E\|$	$\rho(G)$	$\omega(G)$	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time
brock200_1.clq	200	14834	74.543	21	200	≥20	600.00	200	≥20	600.00	200	≥19	600.00	200	≥19	600.00
brock200_2.clq	200	9876	49.628	12	200	12	1.76	200	12	30.56	200	13	137.97	200	≥13	600.00
brock200_3.clq	200	12048	60.543	15	200	15	36.77	200	16	583.05	200	≥15	600.00	200	≥15	600.00
brock200_4.clq	200	13089	65.774	17	200	17	157.28	200	≥17	600.00	200	≥16	600.00	200	≥16	600.00
brock400_1.clq	400	59723	74.841	27	400	≥22	600.00	400	≥20	600.00	400	≥19	600.00	400	≥19	600.00
brock400_2.clq	400	59786	74.920	29	400	≥20	600.00	400	≥20	600.00	400	≥18	600.00	400	≥18	600.00
brock400_3.clq	400	59681	74.788	31	400	≥19	600.00	400	≥20	600.00	400	≥20	600.00	400	≥19	600.00
brock400_4.clq	400	59765	74.894	33	400	≥19	600.00	400	≥20	600.00	400	≥19	600.00	400	≥18	600.00
brock800_1.clq	800	207505	64.927	23	800	≥16	600.00	800	≥16	600.00	800	≥16	600.00	800	≥15	600.00
brock800_2.clq	800	208166	65.133	24	800	≥17	600.00	800	≥17	600.00	800	≥16	600.00	800	≥17	600.00
brock800_3.clq	800	207333	64.873	25	800	≥17	600.00	800	≥17	600.00	800	≥16	600.00	800	≥15	600.00
brock800_4.clq	800	207643	64.970	26	800	≥17	600.00	800	≥16	600.00	800	≥15	600.00	800	≥16	600.00
c-fat200-1.clq	200	1534	7.709	12	200	12	0.01	200	12	0.01	200	12	0.01	200	12	0.01
c-fat200-2.clq	200	3235	16.256	24	200	24	0.01	200	24	0.01	200	24	0.01	200	24	0.01
c-fat200-5.clq	200	8473	42.578	58	200	58	0.01	200	58	0.02	200	58	0.01	200	58	0.02
c-fat500-1.clq	500	4459	3.574	14	500	14	0.01	500	14	0.02	500	14	0.02	500	14	0.02
c-fat500-10.clq	500	46627	37.376	126	500	126	0.03	500	126	0.03	500	126	0.05	500	126	0.08
c-fat500-2.clq	500	9139	7.326	26	500	26	0.01	500	26	0.01	500	26	0.01	500	26	0.02
c-fat500-5.clq	500	23191	18.590	64	500	64	0.01	500	64	0.02	500	64	0.01	500	64	0.02
hamming10-2.clq	1024	518656	99.023	512	1024	512	12.54	1024	512	26.05	1024	512	100.48	1024	512	387.09
hamming10-4.clq	1024	434176	82.894	40	1024	≥17	600.00	1024	≥18	600.00	1024	≥18	600.00	1024	≥15	600.00
hamming6-2.clq	64	1824	90.476	32	64	32	0.01	64	32	0.01	64	32	0.02	64	32	0.01
hamming6-4.clq	64	704	34.921	4	64	4	0.01	64	5	0.02	64	6	0.01	64	6	0.11
hamming8-2.clq	256	31616	96.863	128	256	128	0.08	256	128	0.19	256	128	0.77	256	128	2.29
hamming8-4.clq	256	20864	63.922	16	256	16	0.16	256	16	2.87	256	16	91.39	256	≥15	600.00
johnson16-2-4.clq	120	5460	76.471	8	120	8	32.42	120	9	386.84	120	≥9	600.00	120	≥10	600.00
johnson32-2-4.clq	496	107880	87.879	16	496	≥8	600.00	496	≥7	600.00	496	≥8	600.00	496	≥9	600.00
johnson8-2-4.clq	28	210	55.556	4	28	4	0.01	28	5	0.01	28	5	0.02	28	6	0.02
johnson8-4-4.clq	70	1855	76.812	14	70	14	0.01	70	14	0.02	70	14	0.16	70	15	0.92
keller4.clq	171	9435	64.912	11	171	12	3.84	171	13	65.47	171	≥14	600.00	171	≥15	600.00
keller5.clq	776	225990	75.155	27	776	≥16	600.00	776	≥15	600.00	776	≥15	600.00	776	≥15	600.00
keller6.clq	3361	4619898	81.819	59	3361	≥16	600.00	3361	≥15	600.00	3361	≥15	600.00	3361	≥15	600.00
MANN_a27.clq	378	70551	99.015	126	378	≥21	600.00	378	≥19	600.00	378	≥18	600.00	378	≥18	600.00
MANN_a45.clq	1035	533115	99.630	345	1035	≥21	600.00	1035	≥19	600.00	1035	≥18	600.00	1035	≥18	600.00
MANN_a81.clq	3321	5506380	99.883	1100	3321	≥21	600.00	3321	≥19	600.00	3321	≥18	600.00	3321	≥18	600.00
MANN_a9.clq	45	918	92.727	16	45	17	0.16	45	18	2.34	45	19	13.95	45	20	47.91
p_hat1000-1.clq	1000	122253	24.475	10	1000	11	149.06	1000	≥11	600.00	1000	≥11	600.00	1000	≥11	600.00
p_hat1000-2.clq	1000	244799	49.009	46	1000	≥26	600.00	1000	≥22	600.00	1000	≥22	600.00	1000	≥22	600.00
p_hat1000-3.clq	1000	371746	74.424	68	1000	≥25	600.00	1000	≥25	600.00	1000	≥23	600.00	1000	≥23	600.00
p_hat1500-1.clq	1500	284923	25.343	12	1500	≥12	600.00	1500	≥11	600.00	1500	≥11	600.00	1500	≥11	600.00
p_hat1500-2.clq	1500	568960	50.608	65	1500	≥24	600.00	1500	≥23	600.00	1500	≥22	600.00	1500	≥23	600.00
p_hat1500-3.clq	1500	847244	75.361	94	1500	≥29	600.00	1500	≥26	600.00	1500	≥26	600.00	1500	≥25	600.00

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Table 8 – Continued from previous page

					1-defective clique			2-defective clique			3-defective clique			4-defective clique		
Graph	$\lvert V\rvert$	$\lvert E\rvert$	$\rho(G)$	$\omega(G)$	$\lvert V^{red}\rvert$	opt	time	$\lvert V^{red}\rvert$	opt	time	$\lvert V^{red}\rvert$	opt	time	$\lvert V^{red}\rvert$	opt	time
p_hat300-1.clq	300	10933	24.377	8	300	9	0.30	300	9	4.99	300	10	25.54	300	10	166.98
p_hat300-2.clq	300	21928	48.892	25	300	26	550.26	300	≥22	600.00	300	≥23	600.00	300	≥22	600.00
p_hat300-3.clq	300	33390	74.448	36	300	≥26	600.00	300	≥26	600.00	300	≥24	600.00	300	≥24	600.00
p_hat500-1.clq	500	31569	25.306	9	500	10	4.21	500	11	56.71	500	≥11	600.00	500	≥11	600.00
p_hat500-2.clq	500	62946	50.458	36	500	≥32	600.00	500	≥28	600.00	500	≥27	600.00	500	≥24	600.00
p_hat500-3.clq	500	93800	75.190	50	500	≥30	600.00	500	≥28	600.00	500	≥28	600.00	500	≥28	600.00
p_hat700-1.clq	700	60999	24.933	11	700	12	11.15	700	12	227.75	700	≥11	600.00	700	≥11	600.00
p_hat700-2.clq	700	121728	49.756	44	700	≥26	600.00	700	≥25	600.00	700	≥25	600.00	700	≥23	600.00
p_hat700-3.clq	700	183010	74.805	62	700	≥29	600.00	700	≥28	600.00	700	≥25	600.00	700	≥24	600.00
san1000.clq	1000	250500	50.150	15	1000	≥10	600.00	1000	≥11	600.00	1000	≥11	600.00	1000	≥12	600.00
san200_0.7_1.clq	200	13930	70.000	30	200	≥18	600.00	200	≥18	600.00	200	≥18	600.00	200	≥19	600.00
san200_0.7_2.clq	200	13930	70.000	18	200	≥15	600.00	200	≥15	600.00	200	≥15	600.00	200	≥16	600.00
san200_0.9_1.clq	200	17910	90.000	70	200	≥36	600.00	200	≥34	600.00	200	≥35	600.00	200	≥34	600.00
san200_0.9_2.clq	200	17910	90.000	60	200	≥30	600.00	200	≥28	600.00	200	≥27	600.00	200	≥28	600.00
san200_0.9_3.clq	200	17910	90.000	44	200	≥28	600.00	200	≥26	600.00	200	≥25	600.00	200	≥25	600.00
san400_0.5_1.clq	400	39900	50.000	13	400	≥9	600.00	400	≥10	600.00	400	≥11	600.00	400	≥11	600.00
san400_0.7_1.clq	400	55860	70.000	40	400	≥20	600.00	400	≥20	600.00	400	≥21	600.00	400	≥22	600.00
san400_0.7_2.clq	400	55860	70.000	30	400	≥17	600.00	400	≥18	600.00	400	≥18	600.00	400	≥19	600.00
san400_0.7_3.clq	400	55860	70.000	22	400	≥15	600.00	400	≥16	600.00	400	≥16	600.00	400	≥17	600.00
san400_0.9_1.clq	400	71820	90.000	100	400	≥37	600.00	400	≥30	600.00	400	≥31	600.00	400	≥31	600.00
sanr200_0.7.clq	200	13868	69.688	18	200	≥19	600.00	200	≥18	600.00	200	≥17	600.00	200	≥17	600.00
sanr200_0.9.clq	200	17863	89.764	42	200	≥27	600.00	200	≥28	600.00	200	≥27	600.00	200	≥27	600.00
sanr400_0.5.clq	400	39984	50.105	13	400	14	187.01	400	≥14	600.00	400	≥13	600.00	400	≥13	600.00
sanr400_0.7.clq	400	55869	70.011	21	400	≥21	600.00	400	≥18	600.00	400	≥18	600.00	400	≥18	600.00

Table 9: Detailed results for s-Defective clique and Instances from the 10th DIMACS challenge

					1-defective clique			2-defective clique			3-defective clique			4-defective clique		
Graph	$\|V\|$	$\|E\|$	$\rho(G)$	$\omega(G)$	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time
adjnoun.graph	112	425	6.837	5	89	6	0.01	102	6	0.02	112	7	0.01	112	7	0.03
as-22july06.graph	22963	48436	0.018	17	168	18	0.01	182	18	0.02	204	19	0.25	232	19	0.56
astro-ph.graph	16706	121251	0.087	57	113	57	0.01	113	57	0.01	165	57	0.08	165	57	0.78
caidaRouterLevel.graph	192244	609066	0.003	17	4021	18	1.76	4704	19	6.85	5417	20	67.13	6447	20	286.50
celegans_metabolic.graph	453	2025	1.978	9	92	10	0.01	138	10	0.02	240	11	0.08	313	11	0.31
celegansneural.graph	297	2148	4.887	8	251	8	0.01	265	9	0.05	274	10	0.25	278	10	0.55
chesapeake.graph	39	170	22.942	5	39	6	0.01	39	6	0.01	39	7	0.01	39	7	0.01
cnr-2000.graph	325557	2738969	0.005	84	86	85	0.01	89	85	0.02	170	86	0.02	286	$\geq$80	600.00
coAuthorsCiteseer.graph	227320	814134	0.003	87	87	87	0.01	87	87	0.01	87	87	0.01	87	87	0.01
coAuthorsDBLP.graph	299067	977676	0.002	115	115	115	0.02	115	115	0.03	115	115	0.02	115	115	0.02
cond-mat.graph	16726	47594	0.034	18	18	18	0.01	53	18	0.01	98	18	0.01	164	18	0.02
cond-mat-2003.graph	31163	120029	0.025	25	27	25	0.01	50	25	0.01	77	26	0.06	77	26	0.05
cond-mat-2005.graph	40421	175691	0.022	30	30	30	0.01	30	30	0.02	57	30	0.01	83	30	0.01
dolphins.graph	62	159	8.408	5	45	6	0.01	53	6	0.01	62	6	0.02	62	7	0.01
email.graph	1133	5451	0.850	12	121	12	0.01	238	12	0.03	349	12	0.30	434	13	0.73
football.graph	115	613	9.352	9	115	9	0.01	115	9	0.02	115	9	0.01	115	9	0.01
hep-th.graph	8361	15751	0.045	24	24	24	0.01	24	24	0.01	24	24	0.01	24	24	0.01
jazz.graph	198	2742	14.059	30	30	30	0.01	30	30	0.01	30	30	0.01	30	30	0.01
karate.graph	34	78	13.904	5	22	6	0.01	33	6	0.02	34	6	0.01	34	6	0.01
lesmis.graph	77	254	8.681	10	20	10	0.01	31	11	0.01	38	11	0.01	38	12	0.01
memplus.graph	17758	54196	0.034	97	97	97	0.01	97	97	0.02	97	97	0.01	97	97	0.01
netscience.graph	1589	2742	0.217	20	20	20	0.01	20	20	0.01	20	20	0.01	20	20	0.01
PGPgiantcompo.graph	10680	24316	0.043	25	126	26	0.04	145	27	0.11	171	28	0.16	172	28	0.23
polblogs.graph	1490	16715	1.507	20	459	21	0.37	489	22	3.35	517	22	23.43	541	23	73.80
polbooks.graph	105	441	8.077	6	98	7	0.01	103	7	0.02	105	8	0.01	105	8	0.01
power.graph	4941	6594	0.054	6	36	6	0.01	231	6	0.01	3353	7	0.08	4941	7	0.16
rgg_n_2_17_s0.graph	131072	728474	0.008	15	125	15	0.01	650	16	0.02	2002	16	0.05	6428	16	0.41
rgg_n_2_19_s0.graph	524288	3269220	0.002	18	55	19	0.01	211	19	0.01	534	19	0.01	1995	20	0.03
rgg_n_2_20_s0.graph	1048576	6890866	0.001	17	462	18	0.01	1966	18	0.03	6339	18	0.36	19576	19	3.48

Table 10: Detailed results for s-Defective clique and Instances from the SNAP

					1-defective clique			2-defective clique			3-defective clique			4-defective clique		
Graph	$\|V\|$	$\|E\|$	$\rho(G)$	$\omega(G)$	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time
Cit-HepPh.txt	34546	420877	0.071	19	11284	20	16.80	12471	21	124.58	13697	≥22	600.00	14992	≥21	600.00
Cit-HepTh.txt	27769	352285	0.091	23	7278	24	136.67	7743	≥25	600.00	8167	≥25	600.00	8595	≥25	600.00
Email-EuAll.txt	265009	364481	0.001	16	1852	17	1.67	2026	17	15.18	2227	18	92.95	2470	18	519.33
p2p-Gnutella04.txt	10876	39994	0.068	4	8379	4	1.05	10876	5	1.65	10876	5	3.45	10876	≥6	600.00
p2p-Gnutella24.txt	26518	65369	0.019	4	15519	5	2.39	26518	5	10.02	26518	5	25.96	26518	≥6	600.00
p2p-Gnutella25.txt	22687	54705	0.021	4	13353	5	1.87	22687	5	7.25	22687	5	7.30	22687	≥6	600.00
Slashdot0811.txt	77360	469180	0.016	26	5418	27	69.93	5727	28	588.55	6142	≥7	600.00	6571	≥8	600.00
Slashdot0902.txt	82168	504230	0.015	27	5417	28	34.98	5734	29	244.86	6093	≥8	600.00	6539	≥6	600.00
soc-Epinions1.txt	75879	405740	0.014	23	5243	24	316.18	5456	≥23	600.00	5719	≥22	600.00	6010	≥21	600.00
web-BerkStan.txt	685230	6649470	0.003	201	392	202	0.20	392	202	2.73	392	202	74.77	392	≥162	600.00
web-Google.txt	875713	4322051	0.001	44	218	≥44	600.00	222	≥44	600.00	223	≥45	600.00	223	≥45	600.00
web-NotreDame.txt	325729	1090108	0.002	155	1367	155	4.74	1367	155	331.56	1367	≥152	600.00	1367	≥150	600.00
web-Stanford.txt	281903	1992636	0.005	61	1389	≥59	600.00	1439	≥59	600.00	1499	≥59	600.00	1595	≥36	600.00
Wiki-Vote.txt	7115	100762	0.398	17	2382	18	9.58	2452	19	133.57	2520	≥17	600.00	2604	≥15	600.00

Table 11: Detailed results for s-Defective clique and Instances from the coloring benchmark set

					1-defective clique			2-defective clique			3-defective clique			4-defective clique		
Graph	$\lvert V\rvert$	$\lvert E\rvert$	$\rho(G)$	$\omega(G)$	$\lvert V^{red}\rvert$	opt	time	$\lvert V^{red}\rvert$	opt	time	$\lvert V^{red}\rvert$	opt	time	$\lvert V^{red}\rvert$	opt	time
1-FullIns_3.col	30	100	22.989	3	30	4	0.01	30	5	0.01	30	5	0.01	30	6	0.01
1-FullIns_4.col	93	593	13.862	3	93	4	0.01	93	5	0.01	93	6	0.01	93	6	0.11
1-FullIns_5.col	282	3247	8.195	3	282	4	0.03	282	5	0.06	282	6	0.46	282	6	9.88
1-Insertions_4.col	67	232	10.493	2	67	3	0.01	67	4	0.01	67	4	0.02	67	5	0.02
1-Insertions_5.col	202	1227	6.044	2	202	3	0.02	202	4	0.01	202	4	0.89	202	5	1.75
1-Insertions_6.col	607	6337	3.446	2	607	3	0.09	607	4	0.13	607	4	67.21	607	5	142.24
2-FullIns_3.col	52	201	15.158	4	52	5	0.01	52	5	0.01	52	6	0.01	52	6	0.02
2-FullIns_4.col	212	1621	7.248	4	212	5	0.02	212	6	0.02	212	6	0.06	212	7	0.08
2-FullIns_5.col	852	12201	3.366	4	852	5	0.22	852	6	0.53	852	7	7.78	852	7	9.94
2-Insertions_3.col	37	72	10.811	2	37	3	0.01	37	4	0.01	37	4	0.01	37	4	0.02
2-Insertions_4.col	149	541	4.907	2	149	3	0.01	149	4	0.01	149	4	0.26	149	5	0.34
2-Insertions_5.col	597	3936	2.212	2	597	3	0.03	597	4	0.03	597	4	59.63	597	5	87.05
3-FullIns_3.col	80	346	10.949	5	80	6	0.02	80	6	0.01	80	7	0.01	80	7	0.02
3-FullIns_4.col	405	3524	4.308	5	405	6	0.02	405	7	0.03	405	7	0.31	405	8	0.41
3-FullIns_5.col	2030	33751	1.639	5	2030	6	1.37	2030	7	1.86	2030	8	95.36	2030	8	108.00
3-Insertions_3.col	56	110	7.143	2	56	3	0.01	56	4	0.01	56	4	0.02	56	4	0.01
3-Insertions_4.col	281	1046	2.659	2	281	3	0.02	281	4	0.02	281	4	3.06	281	5	3.73
3-Insertions_5.col	1406	9695	0.982	2	1406	3	0.13	1406	4	0.11	1406	$\geq$4	600.00	1406	$\geq$5	600.00
4-FullIns_3.col	114	541	8.399	6	114	7	0.01	114	7	0.01	114	8	0.01	114	8	0.01
4-FullIns_4.col	690	6650	2.798	6	690	7	0.06	690	8	0.08	690	8	1.16	690	9	1.51
4-FullIns_5.col	4146	77305	0.900	6	4146	7	6.96	4146	8	8.95	4146	$\geq$6	600.00	4146	$\geq$7	600.00
4-Insertions_3.col	79	156	5.063	2	79	3	0.01	79	4	0.01	79	4	0.03	79	4	0.02
4-Insertions_4.col	475	1795	1.594	2	475	3	0.01	475	4	0.01	475	4	24.12	475	5	28.99
5-FullIns_3.col	154	792	6.723	7	136	8	0.01	154	8	0.02	154	9	0.02	154	9	0.02
5-FullIns_4.col	1085	11395	1.938	7	1085	8	0.14	1085	9	0.22	1085	9	3.95	1085	10	4.41
abb313GPIA.col	1557	53356	4.405	8	1552	9	24.93	1552	10	288.96	1555	$\geq$11	600.00	1555	$\geq$12	600.00
anna.col	138	493	5.215	11	19	11	0.01	19	11	0.01	24	12	0.02	44	12	0.01
ash331GPIA.col	662	4181	1.911	3	662	4	0.02	662	4	0.05	662	5	0.13	662	5	139.68
ash608GPIA.col	1216	7844	1.062	3	1216	3	0.03	1216	4	0.09	1216	$\geq$4	600.00	1216	$\geq$5	600.00
ash958GPIA.col	1916	12506	0.682	3	1916	3	0.06	1916	4	0.20	1916	$\geq$4	600.00	1916	$\geq$5	600.00
C2000.5.col	2000	999836	50.017	16	2000	$\geq$13	600.00	2000	$\geq$13	600.00	2000	$\geq$13	600.00	2000	$\geq$13	600.00
C4000.5.col	4000	4000268	50.016	18	4000	$\geq$14	600.00	4000	$\geq$13	600.00	4000	$\geq$13	600.00	4000	$\geq$13	600.00
david.col	87	406	10.853	11	22	11	0.01	33	11	0.01	36	12	0.01	44	12	0.01
DSJC1000.1.col	1000	49629	9.936	6	1000	6	3.32	1000	7	43.42	1000	$\geq$7	600.00	1000	$\geq$7	600.00
DSJC1000.5.col	1000	249826	50.015	15	1000	$\geq$14	600.00	1000	$\geq$13	600.00	1000	$\geq$13	600.00	1000	$\geq$13	600.00
DSJC1000.9.col	1000	449449	89.980	68	1000	$\geq$30	600.00	1000	$\geq$29	600.00	1000	$\geq$27	600.00	1000	$\geq$28	600.00
DSJC125.1.col	125	736	9.497	4	125	5	0.01	125	5	0.02	125	6	0.03	125	6	0.30
DSJC125.5.col	125	3891	50.207	10	125	11	0.25	125	11	1.87	125	12	11.87	125	12	43.24
DSJC125.9.col	125	6961	89.819	34	125	$\geq$27	600.00	125	$\geq$27	600.00	125	$\geq$27	600.00	125	$\geq$26	600.00
DSJC250.1.col	250	3218	10.339	4	250	5	0.02	250	6	0.16	250	6	1.48	250	6	12.84
DSJC250.5.col	250	15668	50.339	12	250	12	14.34	250	13	125.53	250	$\geq$13	600.00	250	$\geq$13	600.00
DSJC250.9.col	250	27897	89.629	43	250	$\geq$29	600.00	250	$\geq$28	600.00	250	$\geq$29	600.00	250	$\geq$28	600.00

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Table 11 – Continued from previous page

Graph	$\lvert V\rvert$	$\lvert E\rvert$	$\rho(G)$	$\omega(G)$	1-defective clique $\lvert V^{red}\rvert$	opt	time	2-defective clique $\lvert V^{red}\rvert$	opt	time	3-defective clique $\lvert V^{red}\rvert$	opt	time	4-defective clique $\lvert V^{red}\rvert$	opt	time
DSJC500.1.col	500	12458	9.986	5	500	6	0.36	500	6	4.34	500	6	41.14	500	7	141.52
DSJC500.5.col	500	62624	50.200	13	500	≥14	600.00	500	≥13	600.00	500	≥13	600.00	500	≥13	600.00
DSJC500.9.col	500	112437	90.130	56	500	≥27	600.00	500	≥26	600.00	500	≥26	600.00	500	≥26	600.00
DSJR500.1.col	500	3555	2.850	11	201	13	0.02	328	14	0.01	423	14	0.01	441	15	0.02
DSJR500.1c.col	500	121275	97.214	83	500	≥35	600.00	500	≥33	600.00	500	≥33	600.00	500	≥34	600.00
DSJR500.5.col	500	58862	47.184	122	488	≥86	600.00	489	≥68	600.00	492	≥66	600.00	492	≥62	600.00
flat1000_50_0.col	1000	245000	49.049	15	1000	≥13	600.00	1000	≥13	600.00	1000	≥13	600.00	1000	≥13	600.00
flat1000_60_0.col	1000	245830	49.215	15	1000	≥14	600.00	1000	≥13	600.00	1000	≥13	600.00	1000	≥13	600.00
flat1000_76_0.col	1000	246708	49.391	15	1000	≥13	600.00	1000	≥13	600.00	1000	≥13	600.00	1000	≥13	600.00
flat300_20_0.col	300	21375	47.659	11	300	12	17.96	300	12	393.64	300	≥12	600.00	300	≥12	600.00
flat300_26_0.col	300	21633	48.234	11	300	12	24.74	300	13	261.79	300	≥13	600.00	300	≥13	600.00
flat300_28_0.col	300	21695	48.372	12	300	12	24.15	300	13	218.12	300	≥12	600.00	300	≥13	600.00
fpsol2.i.1.col	496	11654	9.493	65	85	66	0.01	86	66	0.11	91	66	0.19	120	66	8.99
fpsol2.i.2.col	451	8691	8.565	30	165	31	0.02	203	31	0.11	238	31	0.47	260	31	2.11
fpsol2.i.3.col	425	8688	9.643	30	164	31	0.02	203	31	0.11	238	31	0.72	260	31	2.18
games120.col	120	638	8.936	9	120	9	0.01	120	9	0.01	120	9	0.01	120	9	0.01
homer.col	561	1628	1.036	13	35	13	0.01	61	13	0.01	68	13	0.01	98	13	0.02
huck.col	74	301	11.144	11	20	11	0.02	32	11	0.02	42	11	0.02	45	11	0.01
inithx.i.1.col	864	18707	5.018	54	122	55	1.79	143	55	24.37	150	56	6.37	158	56	16.19
inithx.i.2.col	645	13979	6.731	31	226	31	0.16	278	32	0.38	338	32	4.65	396	32	18.99
inithx.i.3.col	621	13969	7.256	31	212	31	0.11	268	32	0.37	335	32	2.37	396	32	18.25
jean.col	80	254	8.038	10	20	10	0.01	31	11	0.01	38	11	0.02	38	12	0.01
latin_square_10.col	900	307350	75.973	90	900	≥90	600.00	900	≥90	600.00	900	≥90	600.00	900	≥90	600.00
le450_15a.col	450	8168	8.085	15	414	15	0.03	419	15	0.17	420	15	1.16	427	15	3.24
le450_15b.col	450	8169	8.086	15	417	15	0.05	421	15	0.30	427	15	2.20	429	15	7.57
le450_15c.col	450	16680	16.511	15	450	15	0.20	450	15	1.47	450	16	5.63	450	16	35.26
le450_15d.col	450	16750	16.580	15	450	15	0.17	450	15	1.73	450	15	12.20	450	16	54.19
le450_25a.col	450	8260	8.176	25	272	25	0.01	280	25	0.03	289	25	0.17	297	25	0.55
le450_25b.col	450	8263	8.179	25	304	25	0.02	308	25	0.06	314	25	0.31	320	25	1.06
le450_25c.col	450	17343	17.167	25	436	25	0.09	438	25	0.89	439	25	5.07	442	25	22.07
le450_25d.col	450	17425	17.248	25	438	25	0.06	440	25	0.48	441	25	2.61	442	25	11.01
le450_5a.col	450	5714	5.656	5	450	6	0.05	450	6	0.53	450	7	2.34	450	7	6.29
le450_5b.col	450	5734	5.676	5	450	6	0.06	450	6	0.50	450	7	2.64	450	7	6.51
le450_5c.col	450	9803	9.704	5	450	6	0.17	450	7	2.39	450	7	17.36	450	8	70.86
le450_5d.col	450	9757	9.658	5	450	6	0.19	450	7	2.07	450	7	15.66	450	7	80.29
miles1000.col	128	3216	39.567	42	51	43	0.01	61	43	0.02	62	44	0.05	81	44	0.08
miles1500.col	128	5198	63.952	73	84	73	0.19	85	73	0.56	86	74	2.26	88	74	29.87
miles250.col	128	387	4.761	8	27	8	0.01	41	9	0.02	83	9	0.01	102	10	0.01
miles500.col	128	1170	14.395	20	29	21	0.01	35	21	0.01	36	22	0.01	36	22	0.01
miles750.col	128	2113	25.997	31	39	32	0.01	41	33	0.02	43	33	0.02	43	33	0.02
mug100_1.col	100	166	3.354	3	100	4	0.01	100	4	0.01	100	4	0.05	100	5	0.06
mug100_25.col	100	166	3.354	3	100	4	0.02	100	4	0.01	100	4	0.08	100	5	0.05
mug88_1.col	88	146	3.814	3	88	4	0.01	88	4	0.02	88	4	0.03	88	5	0.03

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Table 11 – Continued from previous page

Graph	$\|V\|$	$\|E\|$	$\rho(G)$	$\omega(G)$	1-defective clique $\|V^{red}\|$	1-defective clique opt	1-defective clique time	2-defective clique $\|V^{red}\|$	2-defective clique opt	2-defective clique time	3-defective clique $\|V^{red}\|$	3-defective clique opt	3-defective clique time	4-defective clique $\|V^{red}\|$	4-defective clique opt	4-defective clique time
mug88_25.col	88	146	3.814	3	88	4	0.01	88	4	0.02	88	4	0.03	88	5	0.05
mulsol.i.1.col	197	3925	20.331	49	56	50	0.01	57	50	0.03	63	51	0.05	65	51	0.05
mulsol.i.2.col	188	3885	22.102	31	116	31	0.11	119	32	0.09	122	32	0.13	124	32	0.19
mulsol.i.3.col	184	3916	23.260	31	117	31	0.13	120	32	0.13	123	32	0.14	125	32	0.17
mulsol.i.4.col	185	3946	23.185	31	118	31	0.09	121	32	0.09	124	32	0.14	126	32	0.17
mulsol.i.5.col	186	3973	23.092	31	119	31	0.14	122	32	0.11	125	32	0.13	127	32	0.20
myciel3.col	11	20	36.364	2	11	3	0.01	11	4	0.01	11	4	0.01	11	4	0.02
myciel4.col	23	71	28.063	2	23	3	0.01	23	4	0.01	23	4	0.01	23	5	0.01
myciel5.col	47	236	21.832	2	47	3	0.01	47	4	0.01	47	4	0.02	47	5	0.02
myciel6.col	95	755	16.909	2	95	3	0.01	95	4	0.01	95	4	0.20	95	5	0.47
myciel7.col	191	2360	13.006	2	191	3	0.03	191	4	0.14	191	4	3.01	191	5	9.39
qg.order100.col	10000	990000	1.980	100	10000	≥100	600.00	10000	≥100	600.00	10000	≥100	600.00	10000	≥100	600.00
qg.order30.col	900	26100	6.452	30	900	30	1.53	900	30	35.21	900	30	564.80	900	≥30	600.00
qg.order40.col	1600	62400	4.878	40	1600	40	8.16	1600	40	252.64	1600	≥40	600.00	1600	≥40	600.00
qg.order60.col	3600	212400	3.279	60	3600	60	93.41	3600	≥60	600.00	3600	≥60	600.00	3600	≥60	600.00
queen10_10.col	100	1470	29.697	10	100	10	0.02	100	10	0.06	100	10	0.27	100	10	0.94
queen11_11.col	121	1980	27.273	11	121	11	0.02	121	11	0.13	121	11	0.59	121	11	1.95
queen12_12.col	144	2596	25.214	12	144	12	0.02	144	12	0.19	144	12	1.01	144	12	4.21
queen13_13.col	169	3328	23.443	13	169	13	0.03	169	13	0.33	169	13	1.78	169	13	7.52
queen14_14.col	196	4186	21.905	14	196	14	0.05	196	14	0.56	196	14	3.07	196	14	13.87
queen15_15.col	225	5180	20.556	15	225	15	0.06	225	15	0.81	225	15	5.04	225	15	23.96
queen16_16.col	256	6320	19.363	16	256	16	0.09	256	16	1.26	256	16	8.36	256	16	39.09
queen5_5.col	25	160	53.333	5	25	5	0.01	25	6	0.01	25	6	0.02	25	7	0.01
queen6_6.col	36	290	46.032	6	36	6	0.01	36	6	0.01	36	7	0.01	36	7	0.03
queen7_7.col	49	476	40.476	7	49	7	0.02	49	7	0.01	49	7	0.02	49	8	0.06
queen8_12.col	96	1368	30.000	12	96	12	0.01	96	12	0.05	96	12	0.23	96	12	0.76
queen8_8.col	64	728	36.111	8	64	8	0.01	64	8	0.02	64	8	0.05	64	8	0.16
queen9_9.col	81	1056	32.593	9	81	9	0.01	81	9	0.03	81	9	0.14	81	9	0.42
r1000.1.col	1000	14378	2.878	20	463	21	0.02	665	21	0.05	844	22	0.09	924	22	0.30
r1000.1c.col	1000	485090	97.115	91	1000	≥32	600.00	1000	≥33	600.00	1000	≥33	600.00	1000	≥32	600.00
r1000.5.col	1000	238267	47.701	234	984	≥127	600.00	984	≥86	600.00	985	≥73	600.00	985	≥73	600.00
r125.1.col	125	209	2.697	5	57	6	0.01	101	6	0.01	122	6	0.01	125	7	0.01
r125.1c.col	125	7501	96.787	46	125	≥35	600.00	125	≥36	600.00	125	≥37	600.00	125	≥38	600.00
r125.5.col	125	3838	49.523	36	119	36	0.16	119	37	1.05	120	37	5.01	122	38	12.89
r250.1.col	250	867	2.786	8	70	8	0.02	140	9	0.01	203	9	0.01	234	9	0.01
r250.1c.col	250	30227	97.115	63	250	≥37	600.00	250	≥36	600.00	250	≥37	600.00	250	≥38	600.00
r250.5.col	250	14849	47.708	65	237	65	14.90	237	66	28.33	238	66	333.36	245	≥64	600.00
school1.col	385	19095	25.832	14	361	≥15	600.00	363	≥16	600.00	363	≥17	600.00	363	≥18	600.00
school1_nsh.col	352	14612	23.653	14	331	≥15	600.00	332	≥16	600.00	332	≥17	600.00	333	≥18	600.00
wap01a.col	2368	110871	3.956	41	2107	41	10.72	2166	41	148.51	2281	≥41	600.00	2288	≥40	600.00
wap02a.col	2464	111742	3.682	40	2248	40	14.99	2362	41	148.90	2372	≥41	600.00	2377	≥40	600.00
wap03a.col	4730	286722	2.564	40	4701	≥41	600.00	4702	≥41	600.00	4702	≥40	600.00	4717	≥40	600.00
wap04a.col	5231	294902	2.156	40	5204	41	139.03	5205	≥40	600.00	5207	≥40	600.00	5223	≥40	600.00

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Table 11 – Continued from previous page

Graph	$\|V\|$	$\|E\|$	$\rho(G)$	$\omega(G)$	1-defective clique			2-defective clique			3-defective clique			4-defective clique		
					$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time
wap05a.col	905	43081	10.532	50	675	50	0.42	679	50	4.42	685	50	30.80	693	50	230.77
wap06a.col	947	43571	9.727	40	807	40	3.15	834	40	44.35	846	40	342.30	865	$\geq$40	600.00
wap07a.col	1809	103368	6.321	40	1701	41	12.75	1710	41	248.63	1719	$\geq$42	600.00	1724	$\geq$42	600.00
wap08a.col	1870	104176	5.961	40	1753	40	13.57	1763	40	276.05	1773	$\geq$40	600.00	1779	$\geq$41	600.00
will199GPIA.col	701	6772	2.760	6	700	7	0.03	700	8	0.14	701	8	0.59	701	9	1.39
zeroin.i.1.col	211	4100	18.506	49	73	49	0.14	79	50	0.73	79	50	1.26	91	50	5.82
zeroin.i.2.col	211	3541	15.983	30	106	30	0.02	131	31	0.11	136	31	0.33	137	32	1.14
zeroin.i.3.col	206	3540	16.765	30	106	30	0.03	131	31	0.11	136	31	0.34	137	32	1.11

Table 12: Detailed results for s-Bundle and Instances from the 2nd DIMACS challenge

					2-bundle			3-bundle			4-bundle			5-bundle		
Graph	$\|V\|$	$\|E\|$	$\rho(G)$	$\omega(G)$	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time
brock200_1.clq	200	14834	74.543	21	200	≥21	600.00	200	≥22	600.00	200	≥22	600.00	200	≥21	600.00
brock200_2.clq	200	9876	49.628	12	200	13	78.31	200	≥14	600.00	200	≥15	600.00	200	≥16	600.00
brock200_3.clq	200	12048	60.543	15	200	≥17	600.00	200	≥16	600.00	200	≥17	600.00	200	≥19	600.00
brock200_4.clq	200	13089	65.774	17	200	≥17	600.00	200	≥18	600.00	200	≥18	600.00	200	≥19	600.00
brock400_1.clq	400	59723	74.841	27	400	≥21	600.00	400	≥21	600.00	400	≥22	600.00	400	≥20	600.00
brock400_2.clq	400	59786	74.920	29	400	≥20	600.00	400	≥20	600.00	400	≥22	600.00	400	≥23	600.00
brock400_3.clq	400	59681	74.788	31	400	≥20	600.00	400	≥21	600.00	400	≥22	600.00	400	≥24	600.00
brock400_4.clq	400	59765	74.894	33	400	≥20	600.00	400	≥20	600.00	400	≥21	600.00	400	≥24	600.00
brock800_1.clq	800	207505	64.927	23	800	≥17	600.00	800	≥18	600.00	800	≥19	600.00	800	≥20	600.00
brock800_2.clq	800	208166	65.133	24	800	≥18	600.00	800	≥18	600.00	800	≥17	600.00	800	≥19	600.00
brock800_3.clq	800	207333	64.873	25	800	≥18	600.00	800	≥18	600.00	800	≥18	600.00	800	≥19	600.00
brock800_4.clq	800	207643	64.970	26	800	≥17	600.00	800	≥18	600.00	800	≥19	600.00	800	≥19	600.00
c-fat200-1.clq	200	1534	7.709	12	200	12	0.01	200	12	0.09	200	12	4.65	200	12	301.77
c-fat200-2.clq	200	3235	16.256	24	200	24	0.01	200	24	0.02	200	24	0.62	200	24	12.28
c-fat200-5.clq	200	8473	42.578	58	200	58	0.02	200	58	0.03	200	58	0.34	200	58	3.96
c-fat500-1.clq	500	4459	3.574	14	500	14	0.06	500	14	2.06	500	14	230.48	500	≥8	600.00
c-fat500-10.clq	500	46627	37.376	126	500	126	0.28	500	126	0.38	500	126	2.20	500	126	20.23
c-fat500-2.clq	500	9139	7.326	26	500	26	0.03	500	26	0.50	500	26	23.13	500	≥16	600.00
c-fat500-5.clq	500	23191	18.590	64	500	64	0.03	500	64	0.17	500	64	2.59	500	64	32.65
hamming10-2.clq	1024	518656	99.023	512	1024	≥188	600.00	1024	≥39	600.00	1024	≥30	600.00	1024	≥35	600.00
hamming10-4.clq	1024	434176	82.894	40	1024	≥22	600.00	1024	≥13	600.00	1024	≥13	600.00	1024	≥14	600.00
hamming6-2.clq	64	1824	90.476	32	64	32	0.01	64	32	9.05	64	≥31	600.00	64	≥35	600.00
hamming6-4.clq	64	704	34.921	4	64	6	0.02	64	8	0.05	64	10	1.12	64	12	19.67
hamming8-2.clq	256	31616	96.863	128	256	128	19.10	256	≥39	600.00	256	≥29	600.00	256	≥34	600.00
hamming8-4.clq	256	20864	63.922	16	256	16	87.64	256	≥14	600.00	256	≥13	600.00	256	≥14	600.00
johnson16-2-4.clq	120	5460	76.471	8	120	≥10	600.00	120	≥14	600.00	120	≥16	600.00	120	≥18	600.00
johnson32-2-4.clq	496	107880	87.879	16	496	≥21	600.00	496	≥24	600.00	496	≥25	600.00	496	≥26	600.00
johnson8-2-4.clq	28	210	55.556	4	28	5	0.01	28	8	0.02	28	9	0.77	28	12	1.98
johnson8-4-4.clq	70	1855	76.812	14	70	14	0.20	70	18	328.43	70	≥18	600.00	70	≥20	600.00
keller4.clq	171	9435	64.912	11	171	≥15	600.00	171	≥12	600.00	171	≥15	600.00	171	≥16	600.00
keller5.clq	776	225990	75.155	27	776	≥15	600.00	776	≥12	600.00	776	≥15	600.00	776	≥16	600.00
keller6.clq	3361	4619898	81.819	59	3361	≥15	600.00	3361	≥12	600.00	3361	≥15	600.00	3361	≥16	600.00
MANN_a27.clq	378	70551	99.015	126	378	≥235	600.00	378	≥350	600.00	378	≥350	600.00	378	≥350	600.00
MANN_a45.clq	1035	533115	99.630	345	1035	≥341	600.00	1035	≥338	600.00	1035	≥339	600.00	1035	≥338	600.00
MANN_a81.clq	3321	5506380	99.883	1100	3321	≥333	600.00	3321	≥329	600.00	3321	≥330	600.00	3321	≥329	600.00
MANN_a9.clq	45	918	92.727	16	45	26	1.84	45	36	31.98	45	≥36	600.00	45	45	0.09
p_hat1000-1.clq	1000	122253	24.475	10	1000	≥13	600.00	1000	≥13	600.00	1000	≥13	600.00	1000	≥14	600.00
p_hat1000-2.clq	1000	244799	49.009	46	1000	≥24	600.00	1000	≥22	600.00	1000	≥23	600.00	1000	≥22	600.00
p_hat1000-3.clq	1000	371746	74.424	68	1000	≥25	600.00	1000	≥26	600.00	1000	≥26	600.00	1000	≥28	600.00
p_hat1500-1.clq	1500	284923	25.343	12	1500	≥12	600.00	1500	≥12	600.00	1500	≥13	600.00	1500	≥14	600.00
p_hat1500-2.clq	1500	568960	50.608	65	1500	≥24	600.00	1500	≥26	600.00	1500	≥26	600.00	1500	≥24	600.00
p_hat1500-3.clq	1500	847244	75.361	94	1500	≥27	600.00	1500	≥27	600.00	1500	≥28	600.00	1500	≥29	600.00

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Table 12 – Continued from previous page

Graph	$\|V\|$	$\|E\|$	$\rho(G)$	$\omega(G)$	2-bundle			3-bundle			4-bundle			5-bundle		
					$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time
p_hat300-1.clq	300	10933	24.377	8	300	10	2.79	300	12	464.71	300	≥13	600.00	300	≥13	600.00
p_hat300-2.clq	300	21928	48.892	25	300	≥24	600.00	300	≥23	600.00	300	≥24	600.00	300	≥26	600.00
p_hat300-3.clq	300	33390	74.448	36	300	≥26	600.00	300	≥26	600.00	300	≥25	600.00	300	≥26	600.00
p_hat500-1.clq	500	31569	25.306	9	500	12	64.16	500	≥13	600.00	500	≥12	600.00	500	≥13	600.00
p_hat500-2.clq	500	62946	50.458	36	500	≥28	600.00	500	≥25	600.00	500	≥22	600.00	500	≥23	600.00
p_hat500-3.clq	500	93800	75.190	50	500	≥31	600.00	500	≥28	600.00	500	≥28	600.00	500	≥28	600.00
p_hat700-1.clq	700	60999	24.933	11	700	13	297.66	700	≥12	600.00	700	≥12	600.00	700	≥13	600.00
p_hat700-2.clq	700	121728	49.756	44	700	≥26	600.00	700	≥24	600.00	700	≥21	600.00	700	≥22	600.00
p_hat700-3.clq	700	183010	74.805	62	700	≥27	600.00	700	≥26	600.00	700	≥24	600.00	700	≥24	600.00
san1000.clq	1000	250500	50.150	15	1000	≥16	600.00	1000	≥23	600.00	1000	≥28	600.00	1000	≥31	600.00
san200_0.7_1.clq	200	13930	70.000	30	200	≥28	600.00	200	≥38	600.00	200	≥50	600.00	200	≥63	600.00
san200_0.7_2.clq	200	13930	70.000	18	200	≥23	600.00	200	≥33	600.00	200	≥42	600.00	200	≥48	600.00
san200_0.9_1.clq	200	17910	90.000	70	200	≥59	600.00	200	≥32	600.00	200	≥35	600.00	200	≥38	600.00
san200_0.9_2.clq	200	17910	90.000	60	200	≥41	600.00	200	≥33	600.00	200	≥36	600.00	200	≥38	600.00
san200_0.9_3.clq	200	17910	90.000	44	200	≥30	600.00	200	≥29	600.00	200	≥34	600.00	200	≥37	600.00
san400_0.5_1.clq	400	39900	50.000	13	400	≥14	600.00	400	≥19	600.00	400	≥24	600.00	400	≥29	600.00
san400_0.7_1.clq	400	55860	70.000	40	400	≥34	600.00	400	≥48	600.00	400	≥55	600.00	400	≥55	600.00
san400_0.7_2.clq	400	55860	70.000	30	400	≥25	600.00	400	≥36	600.00	400	≥41	600.00	400	≥32	600.00
san400_0.7_3.clq	400	55860	70.000	22	400	≥22	600.00	400	≥31	600.00	400	≥39	600.00	400	≥37	600.00
san400_0.9_1.clq	400	71820	90.000	100	400	≥55	600.00	400	≥31	600.00	400	≥31	600.00	400	≥35	600.00
sanr200_0.7.clq	200	13868	69.688	18	200	≥19	600.00	200	≥19	600.00	200	≥19	600.00	200	≥21	600.00
sanr200_0.9.clq	200	17863	89.764	42	200	≥29	600.00	200	≥30	600.00	200	≥33	600.00	200	≥36	600.00
sanr400_0.5.clq	400	39984	50.105	13	400	≥14	600.00	400	≥15	600.00	400	≥15	600.00	400	≥16	600.00
sanr400_0.7.clq	400	55869	70.011	21	400	≥19	600.00	400	≥19	600.00	400	≥19	600.00	400	≥21	600.00

Table 13: Detailed results for s-Bundle and Instances from the 10th DIMACS challenge

					2-bundle			3-bundle			4-bundle			5-bundle		
Graph	$\|V\|$	$\|E\|$	$\rho(G)$	$\omega(G)$	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time
adjnoun.graph	112	425	6.837	5	89	6	0.01	102	8	0.05	112	8	1.20	112	10	21.01
as-22july06.graph	22963	48436	0.018	17	168	19	0.13	182	21	5.63	204	22	187.76	232	≥23	600.00
astro-ph.graph	16706	121251	0.087	57	113	57	0.02	113	57	0.02	165	57	2.70	165	57	44.38
caidaRouterLevel.graph	192244	609066	0.003	17	4021	20	33.40	4704	≥7	600.00	5417	≥8	600.00	6447	≥8	600.00
celegans_metabolic.graph	453	2025	1.978	9	92	10	0.01	138	11	0.09	240	12	21.31	313	≥14	600.00
celegansneural.graph	297	2148	4.887	8	251	10	0.03	265	11	1.00	274	12	42.81	278	≥12	600.00
chesapeake.graph	39	170	22.942	5	39	7	0.01	39	8	0.01	39	9	0.03	39	11	0.11
cnr-2000.graph	325557	2738969	0.005	84	86	85	0.05	89	86	0.06	170	86	7.43	286	≥80	600.00
coAuthorsCiteseer.graph	227320	814134	0.003	87	87	87	0.05	87	87	0.06	87	87	0.05	87	87	0.05
coAuthorsDBLP.graph	299067	977676	0.002	115	115	115	0.16	115	115	0.16	115	115	0.16	115	115	0.16
cond-mat.graph	16726	47594	0.034	18	18	18	0.01	53	18	0.02	98	18	0.11	164	18	22.50
cond-mat-2003.graph	31163	120029	0.025	25	27	25	0.02	50	25	0.02	77	26	5.06	77	27	21.50
cond-mat-2005.graph	40421	175691	0.022	30	30	30	0.02	30	30	0.01	57	30	0.02	83	30	3.70
dolphins.graph	62	159	8.408	5	45	6	0.02	53	7	0.02	62	7	0.08	62	9	0.47
email.graph	1133	5451	0.850	12	121	12	0.02	238	12	1.81	349	12	417.46	434	≥12	600.00
football.graph	115	613	9.352	9	115	10	0.01	115	11	0.05	115	12	0.27	115	12	9.95
hep-th.graph	8361	15751	0.045	24	24	24	0.01	24	24	0.01	24	24	0.02	24	24	0.01
jazz.graph	198	2742	14.059	30	30	30	0.01	30	30	0.01	30	30	0.01	30	30	0.01
karate.graph	34	78	13.904	5	22	6	0.01	33	6	0.01	34	8	0.02	34	9	0.05
lesmis.graph	77	254	8.681	10	20	10	0.01	31	11	0.01	38	12	0.02	38	12	0.14
memplus.graph	17758	54196	0.034	97	97	97	0.08	97	97	0.08	97	97	0.08	97	97	0.08
netscience.graph	1589	2742	0.217	20	20	20	0.01	20	20	0.01	20	20	0.01	20	20	0.01
PGPgiantcompo.graph	10680	24316	0.043	25	126	29	0.09	145	31	2.56	171	33	6.01	172	35	11.70
polblogs.graph	1490	16715	1.507	20	459	23	27.35	489	≥26	600.00	517	≥19	600.00	541	≥17	600.00
polbooks.graph	105	441	8.077	6	98	7	0.02	103	9	0.05	105	10	0.95	105	11	18.42
power.graph	4941	6594	0.054	6	36	6	0.02	231	6	1.64	3353	≥6	600.00	4941	≥7	600.00
rgg_n_2_17_s0.graph	131072	728474	0.008	15	125	16	0.01	650	16	66.39	2002	≥16	600.00	6428	≥15	600.00
rgg_n_2_19_s0.graph	524288	3269220	0.002	18	55	19	0.02	211	19	0.73	534	≥19	600.00	1995	≥19	600.00
rgg_n_2_20_s0.graph	1048576	6890866	0.001	17	462	18	0.17	1966	≥19	600.00	6339	≥18	600.00			OoM

Table 14: Detailed results for s-Bundle and Instances from the SNAP

					2-bundle			3-bundle			4-bundle			5-bundle		
Graph	$\|V\|$	$\|E\|$	$\rho(G)$	$\omega(G)$	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time
Cit-HepPh.txt	34546	420877	0.071	19			OoM			OoM			OoM			OoM
Cit-HepTh.txt	27769	352285	0.091	23	7278	≥28	600.00	7743	≥29	600.00	8167	≥29	600.00			OoM
Email-EuAll.txt	265009	364481	0.001	16	1852	19	12.67	2026	≥18	600.00	2227	≥4	600.00	2470	≥5	600.00
p2p-Gnutella04.txt	10876	39994	0.068	4	8379	≥5	600.00			OoM			OoM			OoM
p2p-Gnutella24.txt	26518	65369	0.019	4			OoM			OoM			OoM			OoM
p2p-Gnutella25.txt	22687	54705	0.021	4			OoM			OoM			OoM			OoM
Slashdot0811.txt	77360	469180	0.016	26	5418	≥26	600.00	5727	≥6	600.00	6142	≥7	600.00	6571	≥6	600.00
Slashdot0902.txt	82168	504230	0.015	27	5417	32	598.72	5734	≥6	600.00	6093	≥8	600.00	6539	≥6	600.00
soc-Epinions1.txt	75879	405740	0.014	23	5243	≥25	600.00	5456	≥19	600.00	5719	≥21	600.00	6010	≥21	600.00
web-BerkStan.txt	685230	6649470	0.003	201	392	202	1.58	392	202	6.40	392	202	214.84	392	≥162	600.00
web-Google.txt	875713	4322051	0.001	44	218	≥44	600.00	222	≥45	600.00	223	≥46	600.00	223	≥46	600.00
web-NotreDame.txt	325729	1090108	0.002	155	1367	155	6.86	1367	≥152	600.00	1367	≥150	600.00	1367	≥150	600.00
web-Stanford.txt	281903	1992636	0.005	61	1389	≥59	600.00	1439	≥55	600.00	1499	≥4	600.00	1595	≥5	600.00
Wiki-Vote.txt	7115	100762	0.398	17	2382	21	72.68	2452	≥17	600.00	2520	≥4	600.00	2604	≥5	600.00

Table 15: Detailed results for s-Bundle and Instances from the coloring benchmark set

					2-bundle			3-bundle			4-bundle			5-bundle		
Graph	$\|V\|$	$\|E\|$	$\rho(G)$	$\omega(G)$	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time
1-FullIns_3.col	30	100	22.989	3	30	5	0.01	30	7	0.01	30	8	0.02	30	8	0.17
1-FullIns_4.col	93	593	13.862	3	93	6	0.02	93	7	0.06	93	9	1.64	93	10	29.81
1-FullIns_5.col	282	3247	8.195	3	282	6	0.06	282	8	4.23	282	10	166.95	282	≥10	600.00
1-Insertions_4.col	67	232	10.493	2	67	4	0.01	67	5	0.03	67	6	0.61	67	8	6.13
1-Insertions_5.col	202	1227	6.044	2	202	4	0.02	202	6	0.78	202	8	18.66	202	≥9	600.00
1-Insertions_6.col	607	6337	3.446	2	607	4	0.52	607	6	52.29	607	≥8	600.00	607	≥6	600.00
2-FullIns_3.col	52	201	15.158	4	52	5	0.01	52	7	0.01	52	8	0.09	52	9	0.97
2-FullIns_4.col	212	1621	7.248	4	212	6	0.03	212	8	0.87	212	10	49.97	212	≥10	600.00
2-FullIns_5.col	852	12201	3.366	4	852	7	1.11	852	8	147.98	852	≥8	600.00	852	≥7	600.00
2-Insertions_3.col	37	72	10.811	2	37	4	0.01	37	4	0.01	37	6	0.03	37	7	0.17
2-Insertions_4.col	149	541	4.907	2	149	4	0.02	149	5	0.31	149	6	20.17	149	8	505.33
2-Insertions_5.col	597	3936	2.212	2	597	4	0.44	597	6	53.06	597	≥8	600.00	597	≥8	600.00
3-FullIns_3.col	80	346	10.949	5	80	6	0.01	80	7	0.03	80	8	0.66	80	10	7.36
3-FullIns_4.col	405	3524	4.308	5	405	7	0.13	405	9	10.51	405	≥8	600.00	405	≥10	600.00
3-FullIns_5.col	2030	33751	1.639	5	2030	8	12.89	2030	≥6	600.00	2030	≥6	600.00	2030	≥7	600.00
3-Insertions_3.col	56	110	7.143	2	56	4	0.01	56	4	0.03	56	6	0.19	56	7	1.75
3-Insertions_4.col	281	1046	2.659	2	281	4	0.05	281	5	3.70	281	6	425.46	281	≥8	600.00
3-Insertions_5.col	1406	9695	0.982	2	1406	4	5.60	1406	≥6	600.00	1406	≥7	600.00	1406	≥6	600.00
4-FullIns_3.col	114	541	8.399	6	114	7	0.02	114	8	0.09	114	9	2.17	114	10	55.04
4-FullIns_4.col	690	6650	2.798	6	690	8	0.61	690	10	69.80	690	≥8	600.00	690	≥10	600.00
4-FullIns_5.col	4146	77305	0.900	6	4146	9	117.25	4146	≥6	600.00	4146	≥6	600.00	4146	≥7	600.00
4-Insertions_3.col	79	156	5.063	2	79	4	0.01	79	4	0.06	79	6	0.92	79	7	12.51
4-Insertions_4.col	475	1795	1.594	2	475	4	0.22	475	5	28.33	475	≥6	600.00	475	≥8	600.00
5-FullIns_3.col	154	792	6.723	7	136	8	0.02	154	9	0.31	154	10	8.74	154	11	204.58
5-FullIns_4.col	1085	11395	1.938	7	1085	9	2.12	1085	11	439.55	1085	≥8	600.00	1085	≥7	600.00
abb313GPIA.col	1557	53356	4.405	8	1552	≥14	600.00	1552	≥16	600.00	1555	≥21	600.00	1555	≥23	600.00
anna.col	138	493	5.215	11	19	11	0.02	19	11	0.01	24	12	0.02	44	13	0.52
ash331GPIA.col	662	4181	1.911	3	662	4	0.67	662	6	106.10	662	≥8	600.00	662	≥10	600.00
ash608GPIA.col	1216	7844	1.062	3	1216	4	3.88	1216	≥6	600.00	1216	≥8	600.00	1216	≥10	600.00
ash958GPIA.col	1916	12506	0.682	3	1916	4	14.99	1916	≥6	600.00	1916	≥8	600.00	1916	≥10	600.00
C2000.5.col	2000	999836	50.017	16	2000	≥14	600.00	2000	≥14	600.00	2000	≥14	600.00	2000	≥16	600.00
C4000.5.col	4000	4000268	50.016	18	4000	≥14	600.00	4000	≥14	600.00	4000	≥15	600.00	4000	≥16	600.00
david.col	87	406	10.853	11	22	11	0.01	33	11	0.01	36	12	0.01	44	13	0.13
DSJC1000.1.col	1000	49629	9.936	6	1000	7	5.74	1000	≥8	600.00	1000	≥8	600.00	1000	≥9	600.00
DSJC1000.5.col	1000	249826	50.015	15	1000	≥15	600.00	1000	≥14	600.00	1000	≥15	600.00	1000	≥15	600.00
DSJC1000.9.col	1000	449449	89.980	68	1000	≥30	600.00	1000	≥32	600.00	1000	≥35	600.00	1000	≥38	600.00
DSJC125.1.col	125	736	9.497	4	125	5	0.01	125	7	0.22	125	8	3.96	125	9	111.23
DSJC125.5.col	125	3891	50.207	10	125	13	3.35	125	≥14	600.00	125	≥15	600.00	125	≥15	600.00
DSJC125.9.col	125	6961	89.819	34	125	≥31	600.00	125	≥33	600.00	125	≥36	600.00	125	≥37	600.00
DSJC250.1.col	250	3218	10.339	4	250	6	0.05	250	7	2.90	250	8	229.13	250	≥9	600.00
DSJC250.5.col	250	15668	50.339	12	250	14	573.54	250	≥15	600.00	250	≥16	600.00	250	≥17	600.00
DSJC250.9.col	250	27897	89.629	43	250	≥28	600.00	250	≥27	600.00	250	≥31	600.00	250	≥35	600.00

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Table 15 – Continued from previous page

					2-bundle			3-bundle			4-bundle			5-bundle		
Graph	$\|V\|$	$\|E\|$	$\rho(G)$	$\omega(G)$	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time
DSJC500.1.col	500	12458	9.986	5	500	6	0.69	500	8	54.55	500	≥9	600.00	500	≥9	600.00
DSJC500.5.col	500	62624	50.200	13	500	≥14	600.00	500	≥15	600.00	500	≥15	600.00	500	≥16	600.00
DSJC500.9.col	500	112437	90.130	56	500	≥30	600.00	500	≥30	600.00	500	≥35	600.00	500	≥38	600.00
DSJR500.1.col	500	3555	2.850	11	201	14	0.01	328	15	1.08	423	15	134.10	441	≥11	600.00
DSJR500.1c.col	500	121275	97.214	83	500	≥45	600.00	500	≥53	600.00	500	≥61	600.00	500	≥86	600.00
DSJR500.5.col	500	58862	47.184	122	488	≥62	600.00	489	≥55	600.00	492	≥43	600.00	492	≥38	600.00
flat1000_50_0.col	1000	245000	49.049	15	1000	≥14	600.00	1000	≥14	600.00	1000	≥15	600.00	1000	≥16	600.00
flat1000_60_0.col	1000	245830	49.215	15	1000	≥14	600.00	1000	≥14	600.00	1000	≥15	600.00	1000	≥16	600.00
flat1000_76_0.col	1000	246708	49.391	15	1000	≥14	600.00	1000	≥14	600.00	1000	≥14	600.00	1000	≥15	600.00
flat300_20_0.col	300	21375	47.659	11	300	≥13	600.00	300	≥15	600.00	300	≥14	600.00	300	≥15	600.00
flat300_26_0.col	300	21633	48.234	11	300	≥14	600.00	300	≥14	600.00	300	≥15	600.00	300	≥15	600.00
flat300_28_0.col	300	21695	48.372	12	300	≥13	600.00	300	≥14	600.00	300	≥15	600.00	300	≥16	600.00
fpsol2.i.1.col	496	11654	9.493	65	85	66	1.12	86	66	92.65	91	≥44	600.00	120	≥43	600.00
fpsol2.i.2.col	451	8691	8.565	30	165	31	0.17	203	31	5.07	238	31	127.84	260	≥11	600.00
fpsol2.i.3.col	425	8688	9.643	30	164	31	0.19	203	31	4.90	238	31	128.53	260	≥11	600.00
games120.col	120	638	8.936	9	120	10	0.02	120	10	0.03	120	10	0.62	120	12	16.46
homer.col	561	1628	1.036	13	35	13	0.01	61	13	0.02	68	14	0.20	98	15	10.17
huck.col	74	301	11.144	11	20	11	0.01	32	11	0.01	42	11	0.02	45	11	0.47
inithx.i.1.col	864	18707	5.018	54	122	55	588.92	143	≥34	600.00	150	≥30	600.00	158	≥21	600.00
inithx.i.2.col	645	13979	6.731	31	226	31	1.92	278	32	111.07	338	≥19	600.00	396	≥12	600.00
inithx.i.3.col	621	13969	7.256	31	212	31	1.89	268	32	32.90	335	≥25	600.00	396	≥11	600.00
jean.col	80	254	8.038	10	20	10	0.01	31	11	0.01	38	12	0.01	38	12	0.05
latin_square_10.col	900	307350	75.973	90	900	≥90	600.00	900	≥90	600.00	900	≥90	600.00	900	≥90	600.00
le450_15a.col	450	8168	8.085	15	414	15	0.06	419	15	3.12	420	15	170.10	427	≥11	600.00
le450_15b.col	450	8169	8.086	15	417	15	0.08	421	15	5.44	427	15	400.55	429	≥12	600.00
le450_15c.col	450	16680	16.511	15	450	15	0.56	450	15	54.98	450	≥12	600.00	450	≥12	600.00
le450_15d.col	450	16750	16.580	15	450	15	0.69	450	15	86.64	450	≥12	600.00	450	≥11	600.00
le450_25a.col	450	8260	8.176	25	272	25	0.02	280	25	0.38	289	25	11.33	297	25	372.64
le450_25b.col	450	8263	8.179	25	304	25	0.02	308	25	0.76	314	25	25.69	320	≥25	600.00
le450_25c.col	450	17343	17.167	25	436	25	0.36	438	25	27.16	439	≥25	600.00	442	≥13	600.00
le450_25d.col	450	17425	17.248	25	438	25	0.23	440	25	17.07	441	≥25	600.00	442	≥13	600.00
le450_5a.col	450	5714	5.656	5	450	6	0.14	450	8	10.53	450	≥9	600.00	450	≥8	600.00
le450_5b.col	450	5734	5.676	5	450	6	0.14	450	8	15.85	450	≥8	600.00	450	≥9	600.00
le450_5c.col	450	9803	9.704	5	450	7	0.28	450	8	31.39	450	≥10	600.00	450	≥9	600.00
le450_5d.col	450	9757	9.658	5	450	7	0.25	450	8	34.93	450	≥9	600.00	450	≥9	600.00
miles1000.col	128	3216	39.567	42	51	43	0.05	61	44	1.08	62	45	3.62	81	46	107.62
miles1500.col	128	5198	63.952	73	84	73	11.58	85	73	471.06	86	≥61	600.00	88	≥60	600.00
miles250.col	128	387	4.761	8	27	9	0.01	41	10	0.01	83	10	0.09	102	11	2.36
miles500.col	128	1170	14.395	20	29	21	0.01	35	22	0.02	36	23	0.25	36	24	1.50
miles750.col	128	2113	25.997	31	39	33	0.02	41	33	0.38	43	35	1.06	43	36	1.94
mug100_1.col	100	166	3.354	3	100	4	0.02	100	5	0.06	100	6	2.73	100	7	40.62
mug100_25.col	100	166	3.354	3	100	4	0.01	100	5	0.06	100	6	2.17	100	7	39.69
mug88_1.col	88	146	3.814	3	88	4	0.01	88	5	0.03	88	6	1.45	88	7	21.11

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Table 15 – Continued from previous page

Graph	$\lvert V\rvert$	$\lvert E\rvert$	$\rho(G)$	$\omega(G)$	2-bundle			3-bundle			4-bundle			5-bundle		
					$\lvert V^{red}\rvert$	opt	time	$\lvert V^{red}\rvert$	opt	time	$\lvert V^{red}\rvert$	opt	time	$\lvert V^{red}\rvert$	opt	time
mug88_25.col	88	146	3.814	3	88	4	0.01	88	5	0.03	88	6	1.44	88	7	18.39
mulsol.i.1.col	197	3925	20.331	49	56	50	0.05	57	51	9.53	63	51	101.35	65	52	498.39
mulsol.i.2.col	188	3885	22.102	31	116	31	0.86	119	32	2.26	122	33	18.89	124	34	140.10
mulsol.i.3.col	184	3916	23.260	31	117	31	0.91	120	32	4.87	123	33	48.72	125	34	154.30
mulsol.i.4.col	185	3946	23.185	31	118	31	0.92	121	32	2.47	124	33	52.18	126	34	408.19
mulsol.i.5.col	186	3973	23.092	31	119	31	0.80	122	32	2.62	125	33	56.35	127	34	455.38
myciel3.col	11	20	36.364	2	11	4	0.01	11	5	0.02	11	6	0.01	11	8	0.01
myciel4.col	23	71	28.063	2	23	4	0.01	23	5	0.01	23	6	0.02	23	8	0.06
myciel5.col	47	236	21.832	2	47	4	0.02	47	6	0.02	47	8	0.11	47	9	2.04
myciel6.col	95	755	16.909	2	95	4	0.01	95	6	0.22	95	8	3.89	95	10	68.95
myciel7.col	191	2360	13.006	2	191	4	0.09	191	6	3.70	191	8	111.65	191	≥10	600.00
qg.order100.col	10000	990000	1.980	100			OoM			OoM			OoM			OoM
qg.order30.col	900	26100	6.452	30	900	30	1.95	900	30	421.92	900	≥30	600.00	900	≥30	600.00
qg.order40.col	1600	62400	4.878	40	1600	40	10.78	1600	≥40	600.00	1600	≥40	600.00	1600	≥40	600.00
qg.order60.col	3600	212400	3.279	60	3600	60	116.35	3600	≥60	600.00	3600	≥60	600.00	3600	≥60	600.00
queen10_10.col	100	1470	29.697	10	100	10	0.02	100	10	1.39	100	10	62.69	100	≥13	600.00
queen11_11.col	121	1980	27.273	11	121	11	0.05	121	11	2.42	121	11	118.81	121	≥13	600.00
queen12_12.col	144	2596	25.214	12	144	12	0.06	144	12	4.03	144	12	217.82	144	≥13	600.00
queen13_13.col	169	3328	23.443	13	169	13	0.09	169	13	6.19	169	13	363.64	169	≥13	600.00
queen14_14.col	196	4186	21.905	14	196	14	0.13	196	14	9.77	196	≥14	600.00	196	≥14	600.00
queen15_15.col	225	5180	20.556	15	225	15	0.17	225	15	14.02	225	≥15	600.00	225	≥15	600.00
queen16_16.col	256	6320	19.363	16	256	16	0.23	256	16	21.20	256	≥16	600.00	256	≥16	600.00
queen5_5.col	25	160	53.333	5	25	6	0.01	25	9	0.01	25	10	0.13	25	13	0.33
queen6_6.col	36	290	46.032	6	36	6	0.01	36	9	0.05	36	10	0.89	36	13	5.90
queen7_7.col	49	476	40.476	7	49	7	0.02	49	9	0.14	49	10	3.46	49	13	38.28
queen8_12.col	96	1368	30.000	12	96	12	0.02	96	12	0.94	96	12	38.85	96	≥13	600.00
queen8_8.col	64	728	36.111	8	64	8	0.02	64	9	0.38	64	10	11.02	64	13	172.49
queen9_9.col	81	1056	32.593	9	81	9	0.02	81	9	0.81	81	10	28.43	81	13	578.64
r1000.1.col	1000	14378	2.878	20	463	21	0.06	665	22	8.75	844	≥20	600.00	924	≥12	600.00
r1000.1c.col	1000	485090	97.115	91	1000	≥49	600.00	1000	≥52	600.00	1000	≥66	600.00	1000	≥77	600.00
r1000.5.col	1000	238267	47.701	234	984	≥71	600.00	984	≥69	600.00	985	≥58	600.00	985	≥35	600.00
r125.1.col	125	209	2.697	5	57	6	0.01	101	6	0.03	122	7	2.15	125	8	73.65
r125.1c.col	125	7501	96.787	46	125	≥43	600.00	125	≥58	600.00	125	≥66	600.00	125	≥83	600.00
r125.5.col	125	3838	49.523	36	119	36	4.68	119	38	174.81	120	≥36	600.00	122	≥34	600.00
r250.1.col	250	867	2.786	8	70	8	0.01	140	9	0.17	203	10	21.11	234	≥11	600.00
r250.1c.col	250	30227	97.115	63	250	≥43	600.00	250	≥60	600.00	250	≥70	600.00	250	≥91	600.00
r250.5.col	250	14849	47.708	65	237	65	272.43	237	≥54	600.00	238	≥54	600.00	245	≥32	600.00
school1.col	385	19095	25.832	14	361	≥26	600.00	363	≥28	600.00	363	≥28	600.00	363	≥33	600.00
school1_nsh.col	352	14612	23.653	14	331	≥27	600.00	332	≥32	600.00	332	≥31	600.00	333	≥32	600.00
wap01a.col	2368	110871	3.956	41	2107	41	195.50	2166	≥40	600.00	2281	≥30	600.00	2288	≥28	600.00
wap02a.col	2464	111742	3.682	40	2248	40	236.08	2362	≥40	600.00	2372	≥31	600.00	2377	≥26	600.00
wap03a.col	4730	286722	2.564	40	4701	≥41	600.00	4702	≥40	600.00	4702	≥31	600.00	4717	≥28	600.00
wap04a.col	5231	294902	2.156	40	5204	≥40	600.00	5205	≥40	600.00	5207	≥30	600.00	5223	≥17	600.00

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Table 15 – Continued from previous page

					2-bundle			3-bundle			4-bundle			5-bundle		
Graph	$\|V\|$	$\|E\|$	$\rho(G)$	$\omega(G)$	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time	$\|V^{red}\|$	opt	time
wap05a.col	905	43081	10.532	50	675	50	3.20	679	50	258.56	685	$\geq$32	600.00	693	$\geq$23	600.00
wap06a.col	947	43571	9.727	40	807	40	46.05	834	$\geq$40	600.00	846	$\geq$38	600.00	865	$\geq$23	600.00
wap07a.col	1809	103368	6.321	40	1701	41	119.40	1710	$\geq$42	600.00	1719	$\geq$40	600.00	1724	$\geq$21	600.00
wap08a.col	1870	104176	5.961	40	1753	40	232.29	1763	$\geq$40	600.00	1773	$\geq$40	600.00	1779	$\geq$21	600.00
will199GPIA.col	701	6772	2.760	6	700	8	0.84	700	10	139.54	701	$\geq$12	600.00	701	$\geq$13	600.00
zeroin.i.1.col	211	4100	18.506	49	73	49	40.79	79	$\geq$47	600.00	79	$\geq$35	600.00	91	$\geq$32	600.00
zeroin.i.2.col	211	3541	15.983	30	106	30	0.87	131	31	37.58	136	$\geq$31	600.00	137	$\geq$24	600.00
zeroin.i.3.col	206	3540	16.765	30	106	30	0.87	131	31	36.58	136	$\geq$31	600.00	137	$\geq$24	600.00