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A Comparison of Column-Generation Approaches to the Vehicle Routing Problem with Time Windows and Temporal Synchronized Pickup and Delivery

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Abstract

In the Vehicle Routing Problem with Time Windows and Temporal Synchronized Pickup and Delivery (VRPTWTSPD), user-specified transportation requests from origin to destination points have to be serviced by a fleet of homogeneous vehicles. The task is to find a set of minimum-cost routes satisfying pairing and precedence, capacities, and time windows. Additionally, temporal synchronization constraints couple the service times at the pickup and delivery locations of the customer requests in the following way: A request has to be delivered within a minimum and maximum time lag (called ride time) after it has been picked up. The presence of these ride-time constraints severely complicates the subproblem of the natural column-generation formulation of the VRPTWTSPD. It is not clear if their integration into the subproblem pays off in an integer column-generation approach. We develop four exact approaches to the VRPTWTSPD based on column-generation formulations with differing subproblems. Two of these subproblems are considered for the first time in this paper. We derive new dominance rules and labeling algorithms for their effective solution. Extensive computational results indicate that integrating either both types of ride-time constraints or only the maximum ride-time constraints into the subproblem results in the strongest overall approach.

Key words: pickup and delivery, temporal synchronization, column generation, ride-time constraints, labeling algorithm, branch-and-cut-and-price

1. Introduction

In the family of one-to-one *Pickup-and-Delivery Problems* (PDPs), customer requests consist of transporting goods or people between paired origin and destination points: for each request a specific good or person has to be picked up at one location and to be transported to the corresponding delivery location. Typically, the task is to design a set of minimum-cost routes satisfying all customer requests subject to pairing and precedence, and other problem-specific constraints. For details on different PDP-variants we refer to recent surveys (Berbeglia *et al.*, 2007; Cordeau *et al.*, 2008; Parragh *et al.*, 2008). The qualitative dependence of the visits of paired customer locations, i.e., pick first deliver second, is a key characteristic of PDPs.

A well-studied one-to-one PDP is the *Pickup-and-Delivery Problem with Time Windows* (PDPTW) (e.g., Dumas *et al.*, 1991; Ropke and Cordeau, 2009; Baldacci *et al.*, 2011) in which vehicle routes must respect pairing and precedence, capacities, and time windows. In this article, we introduce the *Vehicle Routing Problem with Time Windows and Temporal Synchronized Pickup and Delivery* (VRPTWTSPD). It extends the PDPTW by imposing additional constraints that couple the service times at the pickup and delivery locations of the customer requests in the following way: A delivery node has to be serviced within a given *minimum* and *maximum* time lag (called *ride time*) after the service at the corresponding pickup node has

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been completed. Because both pickup and delivery are performed by the same vehicle, these additional constraints are temporal *intra-route synchronization* constraints. Compared to other PDP-variants, the synchronization imposes that there is not only a qualitative but also a quantitative dependence of the visiting times of paired customer locations in the VRPTWTSPD. As a generalization of the PDPTW the VRPTWTSPD is clearly $\mathcal{NP}$-hard.

As pointed out, e.g., by Dohn *et al.* (2011) or Drexl (2012), synchronization aspects are highly relevant in routing practice and there is a growing interest in research on *Vehicle Routing Problems* (VRPs) with synchronization constraints. We see the VRPTWTSPD as the prototypical VRP with intra-route synchronization in the sense that synchronization takes place only within disjunctive pairs of nodes and that there are no other non-standard constraints present. In this respect, the development of an effective algorithm for solving the VRPTWTSPD constitutes a central building block for solution approaches to richer VRPs with synchronization constraints.

A special case of the VRPTWTSPD is the so-called *Dial-a-Ride Problem* (DARP) in which only a maximum ride time is specified for each pickup-and-delivery pair. The DARP has been subject to extensive research. For a review on different modeling variants and algorithmic approaches to the DARP we refer to the survey of Cordeau and Laporte (2007). The DARP mainly arises in door-to-door transportation services for school children, handicapped persons, or the elderly and disabled (see, e.g., Russell and Morrel, 1986; Madsen *et al.*, 1995; Toth and Vigo, 1997; Borndörfer *et al.*, 1997). In this context, maximum ride times are used to guarantee a certain service level by limiting the time a passenger is on board of the vehicle. A similar service-related use of maximum ride-time constraints is described by Plum *et al.* (2014) in the context of liner shipping service design.

Other applications of temporal intra-route synchronization in which also a minimum ride time is relevant include the planning of security guards where locations have to be inspected repeatedly within given time intervals (Bredström and Rönnqvist, 2008). There, no actual pickup at one location followed by a delivery at another location takes place. Instead, just a pairing and precedence relation between the services at the nodes forming a customer request is given. Similar planning problems arise in home health care, e.g., when patients have to be monitored by a nurse several times a day (Eveborn *et al.*, 2006). In the service industries, some jobs may require several working steps that cannot be performed in direct succession. A reason might be that after completing one task a workpiece must dry or harden before it can be further processed. Thus, a technician may have to visit a certain location several times with some given time between each visit. Furthermore, when there is a limit on the total working hours of drivers (Ceselli *et al.*, 2009) or when transporting perishable goods (Azi *et al.*, 2010), the time a vehicle is away from the depot has to be restricted. This can be modeled by imposing a maximum ride-time constraint on a dummy request originating and destinating at the depot. Similarly, one might want to have a limit on both the minimum and maximum duration of the routes in order to achieve an even work-distribution of the drivers.

The contributions of this paper are the following: First, we introduce the VRPTWTSPD as the prototypical VRP with temporal intra-route synchronization. This problem has to the best of our knowledge not been considered before. Second, we develop four exact solution approaches to the VRPTWTSPD based on column-generation formulations whose master programs are formulated on different sets of variables implying different subproblems. Two of these subproblems are considered for the first time in this paper. We derive new dominance rules and labeling algorithms for their solution. One of them is the natural subproblem of the VRPTWTSPD, in which time windows as well as temporal intra-route synchronization with both minimum and maximum ride times have to be dealt with. Finally, we report extensive computational results over a large number of test instances with different characteristics regarding the number of customer requests and the tightness of capacity, time-window, and minimum and maximum ride-time constraints.

Integer column-generation methods have proven to be very successful in solving many VRP-variants including PDPs (e.g., Dumas *et al.*, 1991; Ropke and Cordeau, 2009; Baldacci *et al.*, 2011). The column-generation master program of such approaches typically is an extended set-partitioning model formulated on variables representing feasible routes for the problem at hand. These formulations generally provide stronger bounds compared to other formulations like, e.g., arc-flow formulations or extended set-partitioning models formulated on a relaxed set of variables. However, the overall success of an integer column-generation approach for VRP-variants relies not only on strong bounds but also on the effective solution of the subproblem.

This is the main challenge when synchronization comes into play (Drexl, 2012). In the case of inter-route synchronization, additional constraints have to be included in the master programs (Desaulniers *et al.*, 1998). Because of the dual variables associated with these constraints, the resulting subproblems are highly complex (e.g., Christiansen and Nygreen, 1998; Ioachim *et al.*, 1999; Dohn *et al.*, 2011) and cannot be solved by standard dynamic-programming labeling algorithms. This is also true for intra-route synchronization where no additional linking constraints are necessary. There, the increased complexity of the subproblems is not caused by additional duals but by the synchronization constraints themselves, which may be hard to incorporate into the subproblem. For the DARP, e.g., Hunsaker and Savelsbergh (2002) have demonstrated that in the presence of time windows and maximum ride times checking the feasibility of a given route is intricate. Clearly, the effective generation of such routes within a column-generation approach is even more challenging.

In the case of intra-route synchronization, the complexity of the subproblems can be reduced by relaxing one or more types of constraints in the subproblem and handling them in the master programs instead. The resulting easier-to-solve subproblems come at the cost of weaker lower bounds and, thus, larger branch-and-bound trees. Ropke and Cordeau (2005) follow this approach to solve the DARP with column generation. They relax the maximum ride times in the subproblem and enforce them using infeasible path elimination constraints (IPEC) in the master. The resulting subproblem is well studied and effective algorithms for its solution exist (Dumas *et al.*, 1991; Ropke and Cordeau, 2009).

The opposite approach was taken by Gschwind and Irnich (2013) who proposed two branch-and-cut-and-price algorithms for the DARP handling all route constraints in the subproblem. They derived a weak and a strong dominance rule resulting in labeling algorithms with differing effectiveness for the solution of the subproblem. Their computational results indicated that the algorithm based on the weak dominance rule is inferior to the approach of Ropke and Cordeau (2005), while the one using the strong dominance criterion is superior. Thus, it is a priori not clear if the integration of certain route constraints in the subproblem pays off in the overall solution algorithm for a problem. We, therefore, consider and compare the applicability of four column-generation algorithms to the VRPTWTSPD. Each algorithm uses a different subproblem: One that handles all route constraints of the VRPTWTSPD, one that relaxes the minimum ride times, one that relaxes the maximum ride times, and one that relaxes both types of ride-time constraints.

The remainder of the paper is organized as follows. Section 2 defines the VRPTWTSPD and presents column-generation formulations of it. Section 3 describes the dominance rules and labeling algorithms we use for solving the different subproblems. Our basic branch-and-cut-and-price algorithm is briefly described in Section 4. Computational results are reported in Section 5. Section 6 concludes.

2. Problem definition and column-generation formulations

In this section, we give a formal definition of the VRPTWTSPD and describe different column-generation formulations of it.

2.1. Definition of the VRPTWTSPD

The VRPTWTSPD is defined on a directed graph $G = (N, A)$ with node set $N = P \cup D \cup \{0, 2n+1\}$ and arc set A. The subsets $P = \{1, ..., n\}$ and $D = \{n+1, ..., 2n\}$ contain the pickup and delivery nodes of n transportation requests, respectively. Node 0 denotes the origin depot and node $2n+1$ the destination depot. For each request $i = 1, ..., n$, a minimum ride time $\underline{L}_i$ and a maximum ride time $\overline{L}_i$ are specified, coupling the service times at the pickup node i and the delivery node $i+n$.

With each node $i \in N$, a non-negative service duration s_i and a demand d_i such that $d_i = -d_{i+n}$ for all $i = 1, ..., n$ are associated. We assume $d_0 = d_{2n+1} = 0$. Furthermore, a time window $[a_i, b_i]$ in which the service has to be started is associated with each node $i \in N$. When arriving at node i prior to a_i, the vehicle has to wait until time a_i before starting its service. It is also allowed to delay the start of service voluntarily at any node. We assume that there is no restriction on the length of the waiting times. The possibility of delaying the start of service at some nodes is crucial for the feasibility of routes in the presence of ride times and time windows (see Hunsaker and Savelsbergh, 2002; Gschwind and Irnich, 2013).

With each arc $(i, j) \in A$, a routing cost c_{ij} and a travel time t_{ij} are associated. We assume that both routing costs and travel times are non-negative and satisfy the triangle inequality. To serve the n transportation requests, a fleet K of identical vehicles with capacity C is located at the depot 0.

The VRPTWTSPD consists in finding $|K|$ vehicle routes starting and ending at the depot nodes 0 and $2n + 1$, respectively, such that each request is served exactly once and the total routing costs are minimal. Thereby, the routes have to satisfy the following conditions:

pairing and precedence: For each request i, pickup node i and delivery node $i + n$ are visited on the same route, and the pickup node i is visited first.

capacity: The load of the vehicle must not exceed C at any time.

time windows: For each node i, the start of service must lie within the time window $[a_i, b_i]$.

ride times: The service at a delivery node $i + n$ has to start at least $\underline{L}_i$ and at most $\overline{L}_i$ units of time after the service at the corresponding pickup node i has been completed.

Note that it is not straightforward to decide on the feasibility of a route in the VRPTWTSPD sense due to the presence of different types of potentially contrasting temporal constraints. More precisely, to verify the feasibility of a given route $r = (h_1, ..., h_q)$ with $h_1 = 0$ and $h_q = 2n + 1$ one has to find a time schedule $T_r = (\tau_1, ..., \tau_q)$ satisfying

$$\tau_i + s_{h_i} + t_{h_i h_{i+1}} \leq \tau_{i+1} \quad \forall i = 1, ..., q - 1, \tag{1}$$

$$a_{h_i} \leq \tau_i \leq b_{h_i} \quad \forall i = 1, ..., q, \tag{2}$$

$$\tau_i + s_{h_i} + \underline{L}_{h_i} \leq \tau_j \quad \text{if } h_i + n = h_j, \tag{3}$$

$$\tau_i + s_{h_i} + \overline{L}_{h_i} \geq \tau_j \quad \text{if } h_i + n = h_j, \tag{4}$$

where τ_i denotes the start of service at node h_i. Constraints (1) ensure consistency of the service times along the route. Inequalities (2) impose time windows, while (3) and (4) are minimum and maximum ride-time constraints, respectively. A schedule satisfying (1)–(4) is called feasible. We denote by $\mathscr{T}_r$ the set of all feasible schedules for a route r. Furthermore, let $\mathscr{T}_r(t) = \{T_r \in \mathscr{T}_r : \tau_q \leq t\}$ be the set of feasible schedules with start of service τ_q at the last node h_q not later than t.

2.2. Column-generation formulations of the VRPTWTSPD

To formulate the VRPTWTSPD as a set-partitioning problem, let Ω be the set of all VRPTWTSPD-feasible routes. The cost of a route $r \in \Omega$ is denoted by c_r. Moreover, for each route r and each request $i \in P$ denote by $a_{ir} \in \mathbb{Z}$ the number of times request i is performed by route r. Let also λ_r be binary variables indicating if route r is used in the solution. The VRPTWTSPD can then be formulated as follows:

$$(IMP) \quad \min \sum_{r \in \Omega} c_r \lambda_r \tag{5}$$

$$\text{s.t.} \sum_{r \in \Omega} a_{ir} \lambda_r = 1 \quad \forall i \in P, \tag{6}$$

$$\sum_{r \in \Omega} \lambda_r = |K|, \tag{7}$$

$$\lambda_r \in \{0, 1\} \quad \forall r \in \Omega. \tag{8}$$

The objective function (5) minimizes the total routing costs. Partitioning constraints (6) ensure that all requests are served exactly once. Equality (7) imposes the number of routes in the solution, while (8) are binary conditions for route variables.

Typically, the number of feasible routes $|\Omega|$ is very large so that model *IMP* cannot be solved directly. We, therefore, use an integer column-generation approach to solve it. The linear relaxation (*LMP*) of the so-called integer master program *IMP* is initialized with a proper subset of routes and missing routes with negative reduced cost are added dynamically. Integrality is ensured by integrating this process into a branch-and-bound algorithm.

To identify negative reduced-cost routes the column-generation subproblem has to be solved. Let $\pi_i, i \in P$ and μ be the dual variables associated with constraints (6) and (7), respectively. The reduced cost of arc $(i,j) \in A$ is defined as

$$\tilde{c}_{ij} = \begin{cases} c_{ij} - \pi_i & \text{if } i \in P, \\ c_{ij} & \text{otherwise.} \end{cases} \tag{9}$$

The reduced cost $\tilde{c}_r$ of a route $r \in \Omega$ is $\tilde{c}_r = \sum_{(i,j)\in A(r)} \tilde{c}_{ij} - \mu$ where $A(r)$ denotes the sequence of arcs traversed by route r. The subproblem is then given by

$$\min_{r\in\Omega}\{\tilde{c}_r\}. \tag{10}$$

Set-partitioning model *IMP* is the most natural formulation for column-generation based approaches to VRP-variants including the VRPTWTSPD in the following sense: The variable set Ω consists of all routes that are feasible for the problem at hand, i.e., the subproblem takes care of all constraints relating to single routes, while the master program comprises only coupling constraints. Decisive for the success of approaches based on such a formulation is that an effective solution procedure for generating feasible routes with negative reduced cost is available. Because of the simultaneous presence of time-window and ride-time constraints, however, the natural subproblem of the VRPTWTSPD is intricate.

An alternative approach is to formulate the master program in relaxed routing variables $r \in \Omega' \supseteq \Omega$ that may violate one (or more) types of constraints relating to a single route. This can be a promising strategy when generating routes $r \in \Omega$ is complex, while working with the relaxed set Ω' results in a well-solvable subproblem. A commonly used relaxation is to drop the elementarity condition of routes. In this case, the partitioning constraints (6) ensure that non-elementary routes can never be part of an integer solution. Thus, *IMP* with variable set Ω' has the same set of optimal solutions as *IMP* with variable set Ω.

This property, however, does not hold for all relaxations Ω' of Ω and a route $r \in \Omega' \setminus \Omega$ that is infeasible for the original problem might then be part of an integer solution. Consequently, the constraints that have been relaxed in the subproblem must be enforced in the master program to ensure feasibility of the solutions for the original problem. Adding IPEC is one way of doing this. Let $\mathcal{I}$ be the set of all paths that are infeasible with respect to the constraints that have been relaxed in the subproblem. Moreover, for any infeasible path $I \in \mathcal{I}$ and any route r let b_{Ir} be the number of times route r traverses arcs of path I. The IPEC can then be written as

$$\sum_{r\in\Omega'} b_{Ir}\lambda_r \leq |I| - 1 \quad \forall I \in \mathcal{I}, \tag{11}$$

where $|I|$ denotes the length of the infeasible path I, i.e., the number of its arcs. We denote by *IMP-I* a master program that incorporates the set-partitioning model (5)–(8) formulated on a relaxed variable set Ω' together with the IPEC (11) to handle the remaining route constraints.

Obviously, approaches based on formulation *IMP* benefit from stronger LP-bounds compared to approaches using formulation *IMP-I*. The reason is that VRPTWTSPD-infeasible routes, which are excluded in the former, may be convex-combined to form routes that do not violate the IPEC in the latter. Typically, the tighter LP-bounds lead to smaller search trees for *IMP*-based approaches. This comes at the cost of a harder to solve subproblem and it is a priori not clear which formulation enables the overall strongest algorithm.

In the following sections, we consider branch-and-price algorithms for the VRPTWTSPD based on four different column-generation formulations. We denote by IMP^{max}_{min} the approach working on variable set Ω in the master program. The addition of IPEC is not necessary in this case and the master program comprises only coupling constraints. The corresponding subproblem SP^{max}_{min} has to generate VRPTWTSPD-feasible routes.

The other approaches formulate their master programs in routing variables $r \in \Omega'$ that relax either the minimum ride times, the maximum ride times, or both. By *IMP-I* we denote the algorithm that ignores both minimum and maximum ride times in the subproblem (denoted *SP*) and handles them using IPEC in the master program. *SP* is the natural subproblem of the PDPTW and generates routes that respect pairing and precedence, capacity, and time-window constraints, i.e., a time schedule satisfying constraints (1) and (2) exists for such routes.

The approach that handles only the maximum ride times in the subproblem and uses routing variables where the minimum ride times have been relaxed is denoted by $IMP\text{-}I^{max}$. The corresponding subproblem is SP^{max}. It is the natural subproblem of the DARP. Routes generated by SP^{max} satisfy pairing and precedence, and capacity constraints. Moreover, these routes can be assigned a time schedule respecting constraints (1), (2), and (4). $IMP\text{-}I_{min}$ and SP_{min} are the analog to $IMP\text{-}I^{max}$ and SP^{max} where minimum ride rimes are handled in the subproblem.

3. Column-generation subproblems

In this section, we describe solution algorithms for the different subproblems *SP*, SP^{max}, SP_{min}, and SP^{max}_{min}. All four subproblems are Elementary Shortest Path Problems with Resource Constraints (ESPPRC) which are typically solved using dynamic-programming labeling algorithms (Irnich and Desaulniers, 2005). In a labeling algorithm, partial paths are gradually extended in a graph G seeking to find a minimum-cost path from the source node to the sink node. The partial paths are represented by labels that store the accumulated cost and resource consumption along the path. We denote by $\mathcal{P}_\ell$ the partial path corresponding to label ℓ. Decisive for the effectiveness of a labeling algorithm is the use of strong dominance rules to eliminate unpromising labels. A more detailed discussion on ESPPRC and labeling algorithms can be found, e.g., in (Irnich and Desaulniers, 2005).

Note that for the rest of this paper we consider the non-elementary versions of the four subproblems for the following two reasons: First, preliminary computational results indicated that the *LMP* lower bounds obtained by using subproblems with the elementarity conditions were rarely stronger compared to using the corresponding non-elementary subproblems. This resulted in slightly weaker overall algorithms for the former. Second, the extension of all dominance rules and labeling algorithms to the elementary case is straightforward (see Ropke and Cordeau, 2009; Gschwind and Irnich, 2013). In the presence of pairing and precedence, non-elementarity means that a request can be picked up again, after it has been picked up and delivered. Hence, several pickup-and-delivery pairs of the same request can be present in a path. For ease of notation, however, we assume for the rest of the paper that all partial paths are elementary. Furthermore, we assume that the service duration is zero for all nodes. All proofs and argumentations are analog when considering non-elementary partial paths and non-zero service durations.

Also, we assume that the reduced-cost matrix satisfies $\tilde{c}_{ij} \leq \tilde{c}_{ik} + \tilde{c}_{kj}$ for all $(i,j) \in A, k \in D$. Ropke and Cordeau (2009) call this property the *delivery triangle inequality* (DTI). It enables the use of stronger dominance rules for all considered subproblems. Roughly speaking, the DTI ensures that visiting an additional delivery node is never beneficial. Ropke and Cordeau (2009) also show how to transform a reduced-cost matrix that does not satisfy the DTI into one that does, while maintaining the cost of each route unchanged. Hence, working with this assumption is no restriction.

All notation previously introduced for routes is also used for partial paths in the following. Moreover, we use the same notation for all subproblems. The meaning should be clear from the context. The set of feasible schedules $\mathscr{T}_\mathcal{P}$ for a path $\mathcal{P}$, e.g., always refers to feasibility regarding the temporal constraints that are present in the considered subproblem.

3.1. SP - subproblem without ride-time constraints

SP is an (elementary) shortest-path problem with pairing and precedence, capacities, and time windows. It is the natural subproblem of the PDPTW. In this context, it has been subject to prior research (Dumas *et al.*, 1991; Ropke and Cordeau, 2009; Baldacci *et al.*, 2011) and strong dominance rules exist for its solution by a labeling algorithm.

resource	description	SP	SP^{max}	SP_{min}	SP^{max}_{min}
η_ℓ	The node of the label	•	•	•	•
$\tilde{c}_\ell$	The reduced cost	•	•	•	•
t_ℓ	The earliest start of service at the node η_ℓ	•	•	•	•
l_ℓ	The current load	•	•	•	•
O_ℓ	The set of open requests	•	•	•	•
$\tilde{b}_\ell$	The latest feasible start of service at node η_ℓ		•		•
$ld^i_\ell(t)$	The latest possible delivery time of request $i \in O_\ell$		•		•
B^i_ℓ	The point of time when $ld^i_\ell(t)$ becomes constant		•		•
$\underline{ld}^i_\ell(t)$	The latest possible delivery time of request $i \in O_\ell$ such that all other open requests $j \in O_\ell \setminus \{i\}$ are picked up as early as possible				•
$\underline{B}^i_\ell$	The point of time when $\underline{ld}^i_\ell(t)$ becomes constant				•
ed^i_ℓ	The earliest feasible start of service at node η_ℓ			•	•
$\overline{ed}^i_\ell(t)$	The earliest possible delivery time of request $i \in O_\ell$ such that all other open requests $j \in O_\ell \setminus \{i\}$ are picked up as late as possible				•
$\overline{B}^i_\ell$	The point of time when $\overline{ed}^i_\ell(t)$ becomes constant				•

Table 1: Resources of a label ℓ. A bullet indicates that the resource is relevant for the respective subproblem.

In what follows, we summarize the main concepts of Dumas *et al.* (1991) and Ropke and Cordeau (2009) for solving *SP*. Both dominance rule and labeling strategy for *SP* also serve as basis for the solution approaches to the other subproblems in Sections 3.2 - 3.4. Table 1 summarizes all resources that are needed in the solution algorithms for the different subproblems and indicates which resource is relevant for which subproblem.

Dominance rule for SP. Within each label ℓ, the following information has to be stored: the node η_ℓ the label belongs to, its reduced cost $\tilde{c}_\ell$, the earliest start of service t_ℓ at node η_ℓ, and the set of open requests O_ℓ (requests that have been picked up but not yet delivered). Then, the following dominance rule is valid for *SP* (Dumas *et al.*, 1991):

Proposition 1. *(Dom-SP) A feasible label ℓ_1 dominates a label ℓ_2 if*

$$\eta_{\ell_1} = \eta_{\ell_2}, \quad \tilde{c}_{\ell_1} \le \tilde{c}_{\ell_2}, \quad t_{\ell_1} \le t_{\ell_2}, \quad O_{\ell_1} \subseteq O_{\ell_2}. \tag{12}$$

Note that if the DTI does not hold, dominance is only possible between labels with identical sets of open requests $O_{\ell_1} = O_{\ell_2}$.

Labeling algorithm for SP. We now briefly describe the labeling algorithm of Dumas *et al.* (1991) and Ropke and Cordeau (2009) for solving *SP*. In addition to the resources needed for dominance in Proposition 1, they store at each label ℓ the load l_ℓ of the vehicle when leaving η_ℓ, enabling a fast consistency check regarding capacity. The extension of a label ℓ along arc $(\eta_\ell, x) \in A$ is only allowed if either $x \notin O_\ell$ if $x \in P$, or $x-n \in O_\ell$ if $x \in D$, or $O_\ell = \varnothing$ if $x = 2n+1$ holds. Otherwise, pairing and precedence are not satisfied resulting in an infeasible label. Furthermore, consistency with respect to time-window and capacity constraints is ensured by requiring $t_\ell + t_{\eta_\ell,x} \le b_x$ and $l_\ell + d_x \le C$, respectively.

If extending label ℓ along arc $(\eta_\ell, x) \in A$ is feasible, a new label ℓ' is created. Its resources are determined according to the following resource extension functions (REFs):

$$\eta_{\ell'} = x, \quad \tilde{c}_{\ell'} = \tilde{c}_\ell + \tilde{c}_{\eta_\ell,x}, \quad t_{\ell'} = \max\{a_x, t_\ell + t_{\eta_\ell,x}\}, \quad l_{\ell'} = l_\ell + d_x, \tag{13}$$

$$O_{\ell'} = \begin{cases} O_\ell \cup \{x\} & \text{if } x \in P, \\ O_\ell \setminus \{x-n\} & \text{if } x \in D. \end{cases} \tag{14}$$

To reduce the number of labels that have to be processed in the algorithm, unpromising labels are eliminated using dominance rule *Dom-SP*. Moreover, labels that cannot be feasibly completed to node $2n+1$ can be discarded. For *SP*, pairing constraints require that each feasible completion to a label ℓ must visit the delivery nodes $i+n$ of all open requests $i \in O_\ell$ and thereby obey all time-window constraints. If no such completion exists, then label ℓ can be eliminated. When travel times satisfy the triangle inequality, testing if such a completion exists reduces to solving a traveling salesman problem with time windows (TSPTW) over the nodes $\{i+n : i \in O_\ell\} \cup \{\eta_\ell, 2n+1\}$, which is known to be $\mathcal{NP}$-hard. Thus, we consider only subsets of O_ℓ of at most two requests, as proposed by Dumas *et al.* (1991).

3.2. SP^{max} - subproblem with maximum ride-time constraints

In SP^{max}, the natural subproblem of the DARP, paths have to respect pairing and precedence, capacities, time windows, and maximum ride times. The latter two impose that for each feasible path a time schedule satisfying inequalities (1), (2), and (4) must exist. The main difficulty for solution approaches to SP^{max} is to deal with these partially contrasting temporal constraints. In fact, they impose a trade-off between servicing all nodes as early as possible and servicing pickup nodes as late as possible. The implication for labeling algorithms is as follows. Considering only the earliest start of service (as in *Dom-SP*) is not sufficient to guarantee dominance with respect to the temporal constraints of SP^{max} (see Example 1 of Gschwind and Irnich, 2013). Thus, one either has to include additional time-related resources in a dominance rule based on *Dom-SP* or come up with a different strategy to deal with the temporal constraints of SP^{max}.

Recently, Gschwind and Irnich (2013) proposed an effective labeling algorithm for solving SP^{max} that uses an extended version of *Dom-SP* as dominance rule. Their basic idea is the following: Let ℓ be a label with $O_\ell \neq \varnothing$. For each open request $i \in O_\ell$, the corresponding maximum ride-time constraint (4) imposes an upper bound on the start of service at delivery node $i+n$ restricting the set of feasible completions to ℓ. Clearly, a larger value for this bound is preferable. As a result, dominance between two labels is only possible if for each of its open requests the dominating label has a larger upper bound value for the start of service at the respective delivery node. Determining these bounds, however, is not straightforward. They obviously depend on the actual service times at the corresponding pickup nodes within the path $\mathcal{P}_\ell$. Thereby, the possibility to delay the start of service at some nodes has to be incorporated.

Dominance rule for SP^{max}. To formalize their approach in a dominance rule, Gschwind and Irnich (2013) first define the *latest possible delivery time* ld^i_ℓ, i.e., the latest feasible start of service at the delivery node, of an open request $i \in O_\ell$ as a function in the start of service $t \geq t_\ell$ at the current node η_ℓ. Let $\mathcal{P}_\ell = (h_1, ..., h_q = \eta_\ell)$ be the path corresponding to label ℓ. Then, $ld^{h_i}_\ell(t)$ with $t \geq t_\ell$ and $h_i \in O_\ell$ is given by

$$ld^{h_i}_\ell(t) = \min\{b_{h_i+n}, \overline{\tau}_i(t) + \overline{L}_{h_i}\}, \tag{15}$$

where $\overline{\tau}_i(t) = \max_{T_{\mathcal{P}_\ell} \in \mathscr{T}_{\mathcal{P}_\ell}(t)}\{\tau_i\}$ is the latest feasible start of service at the pickup node h_i while $\tau_q \leq t$.

Moreover, two important properties related to $ld^{h_i}_\ell(t)$ are proven. First, they show that the schedule $\overline{T}_{\mathcal{P}_\ell}(t) = (\overline{\tau}_1(t), ..., \overline{\tau}_q(t))$ assigning each node $h_i, i = 1, ..., q$ its latest start of service $\overline{\tau}_i$ is feasible. This means that all open requests and the associated latest delivery times can be treated independently in the dominance criterion. Second, they show that the functions $ld^{h_i}_\ell(t)$ are of the form $ld^{h_i}_\ell(t) = \min\{k^i_1 + t, k^i_2\}$ with constants k^i_1 and k^i_2. With this property, the comparison of two such functions can be simplified to comparing them at two distinct points of time.

Let B^i_ℓ be the point of time when $ld^i_\ell(t)$ becomes constant. The following proposition describes a valid dominance rule for SP^{max} (Gschwind and Irnich, 2013):

Proposition 2. *(Dom-SP^{max}) A feasible label ℓ_1 dominates a label ℓ_2 if*

$$\eta_{\ell_1} = \eta_{\ell_2}, \quad \tilde{c}_{\ell_1} \leq \tilde{c}_{\ell_2}, \quad t_{\ell_1} \leq t_{\ell_2}, \quad O_{\ell_1} \subseteq O_{\ell_2}, \text{ and} \tag{16}$$

$$ld^i_{\ell_1}(t_{\ell_1}) + (t_{\ell_2} - t_{\ell_1}) \geq ld^i_{\ell_2}(t_{\ell_2}) \quad \text{and} \quad ld^i_{\ell_1}(B^i_{\ell_1}) \geq ld^i_{\ell_2}(B^i_{\ell_2}) \quad \forall i \in O_{\ell_1}. \tag{17}$$

Labeling algorithm for SP^{max}. The labeling algorithm of Gschwind and Irnich (2013) for solving SP^{max} is analog to that of Dumas *et al.* (1991) and Ropke and Cordeau (2009) for *SP* sketched in Section 3.1. The additional presence of maximum ride times and the use of dominance rule *Dom-SP*^{max} involve only some minor modifications. First, the extension of a label ℓ along an arc $(\eta_\ell, x) \in A$ is only feasible if $t_\ell + t_{\eta_\ell,x} \leq ld^i_\ell(B^i_\ell)$ holds for all $i \in O_\ell$. Second, the resources $B^i_{\ell'}$, $ld^i_{\ell'}(t_{\ell'})$, and $ld^i_{\ell'}(B^i_{\ell'})$ have to be determined for all $i \in O_{\ell'}$ when creating a new label ℓ' resulting from the extension of ℓ along arc (η_ℓ, x). Gschwind and Irnich (2013) devised the following simple REFs:

$$B^i_{\ell'} = \begin{cases} \min\{\tilde{b}_x, b_{x+n} - \overline{L}_x\} & \text{if } i = x, \\ \max\{t_{\ell'}, \min\{\tilde{b}_x, B^i_\ell + t_{\eta_\ell,x}\}\} & \text{otherwise,} \end{cases} \tag{18}$$

$$ld^i_{\ell'}(t_{\ell'}) = \begin{cases} \min\{b_{x+n}, t_{\ell'} + \overline{L}_x\} & \text{if } i = x, \\ ld^i_\ell(t_\ell) + (\min\{t_{\ell'} - t_{\eta_\ell,x}, B^i_\ell\} - t_\ell) & \text{otherwise,} \end{cases} \tag{19}$$

$$ld^i_{\ell'}(B^i_{\ell'}) = \begin{cases} B^i_{\ell'} + \overline{L}_x & \text{if } i = x, \\ ld^i_\ell(B^i_\ell) - \max\{0, B^i_\ell + t_{\eta_\ell,x} - \tilde{b}_x\} & \text{otherwise,} \end{cases} \tag{20}$$

where

$$\tilde{b}_x = \begin{cases} b_x & \text{if } x \in P, \\ \min\{b_x, ld^{x-n}_\ell(B^{x-n}_\ell)\} & \text{if } x \in D, \end{cases} \tag{21}$$

is the latest feasible start of service at the current node η_ℓ. Third, the information on the latest possible delivery times $ld^i_\ell, i \in O_\ell$ can also be used in the elimination of labels that cannot be feasibly completed to node $2n + 1$.

3.3. SP_{min} - subproblem with minimum ride-time constraints

Subproblem SP_{min} is an (elementary) shortest path problem with pairing and precedence, capacity, time-window, and minimum ride-time constraints. To the best of our knowledge, SP_{min} has not been considered before and an effective labeling algorithm for its solution is presented here for the first time.

Similar to SP^{max}, different types of temporal constraints are present in SP_{min}. More precisely, a schedule satisfying inequalities (1)–(3) must be assignable to each feasible path. The main task for a labeling approach to SP_{min} based on *Dom-SP* is to ensure consistency of the dominance rule with these constraints.

In contrast to SP^{max}, however, the temporal constraint system of SP_{min} is rather straightforward to handle in a labeling algorithm. Both types of constraints that couple the service times at two different nodes are less or equal constraints (from front to back of the path). Consequently, the optimal strategy regarding time-window constraints, i.e., servicing all nodes as early as possible, is also an optimal strategy in the additional presence of minimum ride-time constraints. This implies that waiting and delaying the service at some node is never beneficial and the possibility to do so can be neglected. Still, inequalities (3) induce that a time schedule in SP_{min} is linked not only between consecutive nodes. Thus, for a label ℓ not only the service time at the current node η_ℓ, but also the service times at all open requests $i \in O_\ell$ are important.

Dominance rule for SP_{min}. To obtain a formal dominance criterion for SP_{min}, we follow the approach of Gschwind and Irnich (2013) for SP^{max}. For each open request $i \in O_\ell$ of label ℓ, the minimum ride-time constraints impose a lower bound on the start of service at the delivery node $i + n$. We define this *earliest possible delivery time* $ed^{h_i}_\ell$ for request $h_i \in O_\ell$ as

$$ed^{h_i}_\ell = \max\{a_{h_i+n}, t_\ell + t_{\eta_\ell,h_i+n}, \underline{\tau}_i + \underline{L}_{h_i}\}, \tag{22}$$

where $\underline{\tau}_i = \min_{T_{\mathcal{P}_\ell} \in \mathcal{T}_{\mathcal{P}_\ell}}\{\tau_i\}$ with $\mathcal{P}_\ell = (h_1, ..., h_q = \eta_\ell)$ is the earliest feasible start of service at the pickup node h_i. Regarding the set of feasible completions to ℓ, a small value $ed^{h_i}_\ell$ is obviously less restrictive than a larger one. Furthermore, the following lemma shows that for each feasible path $\mathcal{P}$ the time schedule $\underline{T}_\mathcal{P}$ which assigns each node its earliest feasible start of service is feasible. Thus, the values $ed^{h_i}_\ell$ can be treated independently in a dominance rule.

Lemma 1. *Let $\mathcal{P} = (h_1, ..., h_q)$ be a feasible partial path. Then, $\underline{T}_{\mathcal{P}} = (\underline{\tau}_1, ..., \underline{\tau}_q) \in \mathscr{T}_{\mathcal{P}}$.*

The proofs of all lemmas and propositions are presented in Section A of the Appendix.

Using the values ed^i_ℓ, we obtain the following extension to *Dom-SP* that is a valid dominance criterion for SP_{min}:

Proposition 3. *(Dom-SP_{min}) A feasible label ℓ_1 dominates a label ℓ_2 if*

$$\eta_{\ell_1} = \eta_{\ell_2}, \quad \tilde{c}_{\ell_1} \leq \tilde{c}_{\ell_2}, \quad t_{\ell_1} \leq t_{\ell_2}, \quad O_{\ell_1} \subseteq O_{\ell_2}, \quad \textit{and} \quad ed^i_{\ell_1} \leq ed^i_{\ell_2} \ \forall i \in O_{\ell_1}. \tag{23}$$

Labeling algorithm for SP_{min}. SP_{min} can be solved using the labeling algorithm of Section 3.1 for solving *SP*. Let ℓ' be the label resulting from the extension of label ℓ along arc (η_ℓ, x). The REFs for the additional resources $ed^i_{\ell'}, i \in O_{\ell'}$ are

$$ed^i_{\ell'} = \begin{cases} \max\{a_{x+n}, t_{\ell'} + t_{x,x+n}, t_{\ell'} + \underline{L}_x\} & \text{if } i = x, \\ \max\{ed^i_\ell, t_{\ell'} + t_{x,i+n}\} & \text{otherwise.} \end{cases} \tag{24}$$

Moreover, the REF (13) for the earliest start of service t_ℓ has to be replaced by

$$t_{\ell'} = \begin{cases} \max\{ed^{x-n}_\ell, t_\ell + t_{\eta_\ell,x}\} & \text{if } x \in D, \\ \max\{a_x, t_\ell + t_{\eta_\ell,x}\} & \text{otherwise.} \end{cases} \tag{25}$$

Again, the information ed^i_ℓ is also used for eliminating labels that cannot be completed to feasible $(0; 2n+1)$-paths.

3.4. SP^{max}_{min} - subproblem with minimum and maximum ride-time constraints

Subproblem SP^{max}_{min} is the natural subproblem of the VRPTWTSPD in which generated paths represent VRPTWTSPD-feasible routes, i.e., they have to respect pairing and precedence, capacities, time windows, and minimum and maximum ride times. The implied scheduling problem is (1)–(4). It simultaneously includes both minimum and maximum ride times which significantly complicates SP^{max}_{min} compared to SP^{max} and SP_{min}. The key problem is the interference of different types of ride-time constraints of different requests so that a straightforward combination of the approaches of Sections 3.2 and 3.3 for SP^{max} and SP_{min}, respectively, is not possible. We demonstrate this in more detail in the following.

Generalizing the idea of Gschwind and Irnich (2013), the minimum and maximum ride times of an open request $i \in O_\ell$ impose a lower bound (ed^i_ℓ) and an upper bound (ld^i_ℓ), respectively, on the start of service at the delivery node $i + n$. Again, a small value ed^i_ℓ and a large value ld^i_ℓ are preferable. The implied optimal strategies for the start of service at the pickup node i, i.e., an early-as-possible service to minimize ed^i_ℓ and a late-as-possible service to maximize ld^i_ℓ, are clearly opposing. Even more, different strategies for the pickup times of different open requests may interfere. More precisely, servicing one node late may imply that another one cannot be serviced early, and vice versa. As a result, there is generally no feasible time schedule that minimizes the values ed^i_ℓ for some $i \in O_\ell$ and at the same time maximizes the values ld^i_ℓ for some other $i \in O_\ell$. Thus in SP^{max}_{min}, the open requests and the associated earliest and latest delivery times cannot be treated independently in a dominance rule. Table 2 gives a small example to illustrate this.

Let ℓ_1 and ℓ_2 be two labels representing the paths $\mathcal{P}_{\ell_1} = (0, i, j, k)$ and $\mathcal{P}_{\ell_2} = (0, j, i, k)$, respectively. Assume identical travel times of 10 between all nodes. Furthermore, let the minimum and maximum ride times for all requests be 40 and 50, respectively. The time windows of nodes 0, i, j, and k are specified in Table 2, while the time windows at the corresponding delivery nodes are assumed to be not binding ($[0, \infty]$). Then, the earliest possible delivery times of requests i and j for label ℓ_1 (as defined in Section 3.3) are $ed^i_{\ell_1} = \max\{0, 50 + 10, 10 + 40\} = 60$ and $ed^j_{\ell_1} = \max\{0, 50 + 10, 20 + 40\} = 60$. The latest possible delivery times $ld^i_{\ell_1}$ and $ld^j_{\ell_1}$ as defined in Section 3.2 are, in general, functions in the start of service at the current node $\eta_{\ell_1} = k$. Here, the only feasible start of service at node k is at time 50. Thus, we only need to consider

Label ℓ_1 for path $(0, i, j, k)$	nodes in $\mathcal{P}(\ell_1)$	0	i	j	k
	time window $[a_., b_.]$	$[0, 100]$	$[0, 30]$	$[20, 50]$	$[50, 50]$
	earliest start of service t_{ℓ_1}	0	10	20	50
	earliest possible delivery $ed^{\cdot}_{\ell_1}$	–	60	60	90
	latest possible delivery $ld^{\cdot}_{\ell_1}$	–	80	90	100

Label ℓ_2 for path $(0, j, i, k)$	nodes in $\mathcal{P}(\ell_2)$	0	j	i	k
	time window $[a_., b_.]$	$[0, 100]$	$[20, 50]$	$[0, 30]$	$[50, 50]$
	earliest start of service t_{ℓ_2}	0	20	30	50
	earliest possible delivery $ed^{\cdot}_{\ell_2}$	–	60	70	90
	latest possible delivery $ld^{\cdot}_{\ell_2}$	–	70	80	100

Table 2: Label ℓ_1 dominates label ℓ_2 in the sense of $Dom - SP^{max}$ and $Dom - SP_{min}$. This does not imply a valid dominance relation for SP^{max}_{min}.

the values $ld^i_{\ell_1}(50) = \min\{\infty, 30 + 50\} = 80$ and $ld^i_{\ell_1}(50) = \min\{\infty, 40 + 50\} = 90$, which imply delaying the start of service at node i until time 30 and at node j until time 40. It is easy to see, however, that there is no feasible time schedule $T_{\mathcal{P}_{\ell_1}}$ that at the same time allows a latest possible delivery time of 80 for request i implying a service time not smaller than 30 at node i and an earliest possible delivery time of 60 for request j implying a service time not larger than 20 at node j.

As a result, simply combining the relations for ed^i_ℓ and ld^i_ℓ of dominance rules *Dom-*SP_{min} and *Dom-*SP^{max} does not lead to a valid dominance criterion for SP^{max}_{min}. The completion $Q = (j+n, i+n, k+n, 2n+1)$, e.g., is feasible for label ℓ_2 but infeasible for ℓ_1, although $ed^x_{\ell_1} < ed^x_{\ell_2}$ and $ld^x_{\ell_1}(50) > ld^x_{\ell_2}(50)$ hold for $x = i, j, k$ (see Table 2).

Dominance rule for SP^{max}_{min}. The example above has shown that the interdependence of different open requests has to be incorporated when trying to dominate labels in SP^{max}_{min}. Roughly speaking, this means that one has to be careful when determining the earliest and latest delivery times of the open requests of a label.

On the one hand, there are completions where one or more requests can only be delivered at the earliest or latest possible time. Consequently, for a dominated label ℓ we have to consider the best possible values for feasible delivery times of all open requests $i \in O_\ell$, i.e., we consider ed^i_ℓ and $ld^i_\ell(t), t \geq t_\ell$ as defined in Sections 3.3 and 3.2, respectively. Note again that here $\mathscr{T}_{\mathcal{P}_\ell}$ refers to the set of all schedules satisfying constraint system (1)–(4).

On the other hand, a completion might generally require picking up some open requests early and some other open requests late. For a dominating label ℓ with $\mathcal{P}_\ell = (h_1, ...h_q = \eta_\ell)$ we, therefore, determine two bounds on the service time of an open request $h_i \in O_\ell$ that are independent of the service times at the other open requests $h_j \in O_\ell \setminus \{h_i\}$.

First, we use the following upper bound $\underline{\tau}^{\overline{O}_\ell}_i(t)$ for an early-as-possible service at a pickup node h_i within path $\mathcal{P}_\ell$. Practically speaking, $\underline{\tau}^{\overline{O}_\ell}_i(t)$ gives the earliest service time at h_i that can be attained without restricting the pickup times of the other open requests. Or from the opposite perspective, when scheduling all other open requests $h_j \in O_\ell \setminus \{h_i\}$ in the most unfavorable way for picking up h_i early, i.e., as late as possible, then the earliest possible service time at h_i that is still feasible is $\underline{\tau}^{\overline{O}_\ell}_i(t)$. Formally, $\underline{\tau}^{\overline{O}_\ell}_i(t) = \min_{T_{\mathcal{P}_\ell} \in \mathscr{T}^{i,\overline{O}_\ell}_{\mathcal{P}_\ell}(t)} \{\tau_i\}$ with $\mathscr{T}^{i,\overline{O}_\ell}_{\mathcal{P}_\ell}(t) = \{T_{\mathcal{P}_\ell} \in \mathscr{T}_{\mathcal{P}_\ell}(t) : \tau_j \geq \overline{\tau}_j(t)\ \forall h_j \in O_\ell \setminus \{h_i\}\}$. Note that $\underline{\tau}^{\overline{O}_\ell}_i(t)$ is a function in t, as the times $\overline{\tau}_j(t)$ depend on t.

Second and analog to $\underline{\tau}^{\overline{O}_\ell}_i(t)$, a lower bound for a late-as-possible service at node h_i is denoted by $\overline{\tau}^{\underline{O}_\ell}_i(t)$. It gives the latest feasible start of service at the pickup node h_i such that the start of service at all other open requests $h_j \in O_\ell \setminus \{h_i\}$ takes its minimal value $\underline{\tau}_j$ and $\tau_q \leq t$, i.e., $\overline{\tau}^{\underline{O}_\ell}_i(t) = \max_{T_{\mathcal{P}_\ell} \in \mathscr{T}^{i,\underline{O}_\ell}_{\mathcal{P}_\ell}(t)} \{\tau_i\}$ with $\mathscr{T}^{i,\underline{O}_\ell}_{\mathcal{P}_\ell}(t) = \{T_{\mathcal{P}_\ell} \in \mathscr{T}_{\mathcal{P}_\ell}(t) : \tau_j \leq \underline{\tau}_j\ \forall h_j \in O_\ell \setminus \{h_i\}\}$. Maximizing the service at h_i may delay the start of service τ_q at the current node η_ℓ. Thus, $\overline{\tau}_i^{O_\ell}(t)$ is also a function in t.

Using $\underline{\tau}_i^{O_\ell}(t)$ and $\overline{\tau}_i^{O_\ell}(t)$ we have the following upper and lower bounds for the earliest and latest delivery times of an open request $h_i \in O$, respectively:

$$\overline{ed}_\ell^{h_i}(t) = \max\{a_{h_i+n}, t_\ell + t_{\eta_\ell, h_i+n}, \underline{\tau}_i^{O_\ell}(t) + \underline{L}_{h_i}\}, \tag{26}$$

$$\underline{ld}_\ell^{h_i}(t) = \min\{b_{h_i+n}, \overline{\tau}_i^{O_\ell}(t) + \overline{L}_{h_i}\}. \tag{27}$$

A valid dominance rule for SP_{min}^{max} is then given by:

Proposition 4. *(Dom*-SP_{min}^{max}) A feasible label ℓ_1 dominates a label ℓ_2 if*

$$\eta_{\ell_1} = \eta_{\ell_2}, \quad \tilde{c}_{\ell_1} \leq \tilde{c}_{\ell_2}, \quad t_{\ell_1} \leq t_{\ell_2}, \quad O_{\ell_1} \subseteq O_{\ell_2}, \text{ and} \tag{28}$$

$$\overline{ed}_{\ell_1}^i(t) \leq ed_{\ell_2}^i \quad \text{and} \quad \underline{ld}_{\ell_1}^i(t) \geq ld_{\ell_2}^i(t) \quad \forall i \in O_{\ell_1}, t \in [t_{\ell_2}, \tilde{b}_{\ell_2}]. \tag{29}$$

Applying dominance rule *Dom*-SP_{min}^{max}* requires the comparison of different functions in inequalities (29) which is clearly not practicable within a labeling algorithm. The following lemma helps characterizing the shape of the functions $\overline{ed}_\ell^i(t)$ and $\underline{ld}_\ell^i(t)$ allowing for a simplified version of *Dom*-SP_{min}^{max}*.

Lemma 2. *Let $\mathcal{X}$ be the set of all $(x_1, ..., x_q) \in \mathbb{R}^q$ satisfying*

$$a_i \leq x_i \leq b_i \quad \forall i = 1, ..., q, \tag{30}$$

$$x_i + c_{ij} \leq x_j \quad \forall j = 2, ..., q; i < j, \tag{31}$$

$$x_i + d_{ij} \geq x_j \quad \forall j = 2, ..., q; i < j, \tag{32}$$

with real-valued constants a_i, b_i, c_{ij}, and d_{ij}. Denote $\mathcal{X}(t) = \{X \in \mathcal{X} : x_q \leq t\}, t \in \mathbb{R}$. Let also be $\overline{x}_i = \max_{X\in\mathcal{X}}\{x_i\}$, $\underline{x}_i = \min_{X\in\mathcal{X}}\{x_i\}$, $\overline{x}_i(t) = \max_{X\in\mathcal{X}(t)}\{x_i\}$, and $\underline{x}_i(t) = \min_{X\in\mathcal{X}(t)}\{x_i\}$. Furthermore, define by $\mathcal{X}^{\underline{S}}(t) = \{X \in \mathcal{X}(t) : x_i \leq \underline{x}_i(t)\ \forall i \in \underline{S}\}$ and by $\mathcal{X}^{\overline{S}}(t) = \{X \in \mathcal{X}(t) : x_i \geq \overline{x}_i(t)\ \forall i \in \overline{S}\}$ with $\underline{S}, \overline{S} \subseteq \{1, ..., q\}$. Denote $\overline{x}_i^{\underline{S}}(t) = \max_{X\in\mathcal{X}^{\underline{S}}(t)}\{x_i\}$ and $\underline{x}_i^{\overline{S}}(t) = \min_{X\in\mathcal{X}^{\overline{S}}(t)}\{x_i\}$. Finally, let t^* be the smallest t with $\mathcal{X}(t) \neq \varnothing$. Then, the following properties hold:

1. *$\overline{x}_i(t) = \min\{k_i^1, k_i^2 + t\}$ for all $i = 1, ..., q, t \geq t^*$ with constants k_i^1 and k_i^2.*
2. *$\underline{x}_i(t) = \underline{x}_i$ for all $i = 1, ..., q, t \geq t^*$.*
3. *$\overline{x}_i^{\underline{S}}(t) = \min\{k_i^1, k_i^2 + t\}$ for all $i = 1, ..., q, t \geq t^*$ with constants k_i^1 and k_i^2.*
4. *$\underline{x}_i^{\overline{S}}(t) = \max\{k_i^1, \min\{k_i^2, k_i^3 + t\}\}$ for all $i = 1, ..., q, t \geq t^*$ with constants k_i^1, k_i^2, and k_i^3.*

Clearly, the scheduling problem (1)–(4) of SP_{min}^{max} is a special case of the constraint system (30)–(32) considered in Lemma 2. Thus, the functions $\overline{ed}_\ell^i(t)$, $ld_\ell^i(t)$, and $\underline{ld}_\ell^i(t)$ are of the forms $\overline{ed}_\ell^i(t) = \max\{k_i^1, \min\{k_i^2, k_i^3 + t\}\}$, $ld_\ell^i(t) = \min\{k_i^4, k_i^5 + t\}$, and $\underline{ld}_\ell^i(t) = \min\{k_i^6, k_i^7 + t\}$, where all k_i are constants. This result is similar to that of Gschwind and Irnich (2013) for the latest possible delivery times in SP^{max}. With it, the comparison of the functions in *Dom*-SP_{min}^{max}* can be reduced to comparing them at two distinct points of time. Denote by $\underline{B}_\ell^i$ and $\overline{B}_\ell^i$ the points of time such that $\underline{ld}_\ell^i(t) = \underline{ld}_\ell^i(\underline{B}_\ell^i)$ and $\overline{ed}_\ell^i(t') = \overline{ed}_\ell^i(\overline{B}_\ell^i)$ holds for all $t \geq \underline{B}_\ell^i$ and $t' \geq \overline{B}_\ell^i$, respectively. Then, the following dominance rule for SP_{min}^{max} results:

Proposition 5. *(Dom-SP_{min}^{max}) A feasible label ℓ_1 dominates a label ℓ_2 if*

$$\eta_{\ell_1} = \eta_{\ell_2}, \quad \tilde{c}_{\ell_1} \leq \tilde{c}_{\ell_2}, \quad t_{\ell_1} \leq t_{\ell_2}, \quad O_{\ell_1} \subseteq O_{\ell_2}, \tag{33}$$

$$\overline{ed}_{\ell_1}^i(t_{\ell_1}) \leq ed_{\ell_2}^i \quad \text{and} \quad \overline{ed}_{\ell_1}^i(\overline{B}_{\ell_1}^i) - \max\{0, \overline{B}_{\ell_1}^i - \tilde{b}_{\ell_2}\} \leq ed_{\ell_2}^i \quad \forall i \in O_{\ell_1}, \text{ and} \tag{34}$$

$$\underline{ld}_{\ell_1}^i(t_{\ell_1}) + (t_{\ell_2} - t_{\ell_1}) \geq ld_{\ell_2}^i(t_{\ell_2}) \quad \text{and} \quad \underline{ld}_{\ell_2}^i(\underline{B}_{\ell_1}^i) \geq ld_{\ell_2}^i(B_{\ell_2}^i) \quad \forall i \in O_{\ell_1}. \tag{35}$$

Note that for determining valid bounds $\overline{ed}_\ell^{h_i}$ and $\underline{ld}_\ell^{h_i}$ on the earliest and latest delivery times of request h_i that do not restrict the pickup times of other open requests $h_j \in O_\ell \setminus \{h_i\}$, it is generally not necessary to consider the maximum and minimum values $\overline{\tau}_j$ and $\underline{\tau}_j$ for the starts of service at the nodes h_j. Instead, it is sufficient to ensure that all h_j can be delivered at their earliest or latest possible delivery times $ed_\ell^{h_j}$ and $ld_\ell^{h_j}$. These times induce starting the service not later than $ed_\ell^{h_j} - \underline{L}_{h_j} \geq \underline{\tau}_j$ and not earlier than $ld_\ell^{h_j} - \underline{L}_{h_j} \leq \overline{\tau}_j$. Hence, bounds that are stronger than $\overline{ed}_\ell^i$ and $\underline{ld}_\ell^i$ can be obtained enabling more dominance when used in *Dom-*SP_{min}^{max}. For simplicity of notation and exposition this has been disregarded in the derivation of *Dom-*SP_{min}^{max}. All proofs, however, are analog.

Dom-SP_{min}^{max} can further be strengthened by using a concept proposed by Gschwind and Irnich (2013) for *Dom-*SP^{max}. Let ℓ be the parent label of ℓ'. The information ed_ℓ^i and ld_ℓ^i on the feasible delivery times of open requests $i \in O_\ell$ can be used to determine an upper bound on the start of service at node $\eta_{\ell'}$ for which $\mathcal{P}_{\ell'}$ can be completed to a feasible $(0; 2n+1)$-path. Using this bound, which is generally smaller than $\tilde{b}_{\ell'}$, strengthens the dominance relation in *Dom-*SP_{min}^{max} (see Section 4.6 in Gschwind and Irnich, 2013, for details).

Labeling algorithm for SP_{min}^{max}. The basic course of our labeling algorithm with *Dom-*SP_{min}^{max} for solving SP_{min}^{max} is identical to those in Sections 3.1-3.3. When creating a new label ℓ', the resources $\eta_{\ell'}$, $\tilde{c}_{\ell'}$, $l_{\ell'}$, and $O_{\ell'}$ are updated using the REFs (13) and (14). The earliest start of service $t_{\ell'}$ is set according to the adapted REF (25).

Determining the values of the resources related to feasible delivery times of open requests $i \in O_{\ell'}$ is intricate. Because of the simultaneous handling of minimum and maximum ride-time constraints, the information on the earliest and latest delivery is interdependent. As a consequence, the determination of these values is much more complex than in the isolated cases in SP_{min} and SP^{max}. The key problems are the following: When creating a new label ℓ', the implied scheduling problem has additional constraints compared to the scheduling problem implied by the parent label ℓ. These constraints impose bounds on the start of service at the current node $\eta_{\ell'}$ that may restrict other service times within the schedule. The impacts on these service times may further propagate throughout the constraint system (see proofs of Proposition 4 and Lemma 2) so that their effect on the earliest and latest pickup times at the open requests $i \in O_{\ell'}$ is non-trivial to identify. Moreover, if the extended label ℓ' ends at a delivery node $\eta_{\ell'} \in D$, then the corresponding request $\eta_\ell - n$ is no longer open. For all open requests $h_i \in O_{\ell'}$, this reduces the set of requests whose latest and earliest service times have to be taken into account when determining the bounds $\underline{\tau}_i^{O_{\ell'}}$ and $\overline{\tau}_i^{O_{\ell'}}$, respectively. Thus, the relation between the resource values $\overline{ed}_{\ell'}^i(t_{\ell'})$, $\overline{ed}_{\ell'}^i(\overline{B}_{\ell'}^i)$, $\underline{ld}_{\ell'}^i(t_{\ell'})$, and $\underline{ld}_{\ell'}^i(\underline{B}_{\ell'}^i)$ with $i \in O_{\ell'}$ and the corresponding values $\overline{ed}_\ell^i(t_\ell)$, $\overline{ed}_\ell^i(\overline{B}_\ell^i)$, $\underline{ld}_\ell^i(t_\ell)$, and $\underline{ld}_\ell^i(\underline{B}_\ell^i)$ of the parent label ℓ is highly complex.

As a result, we were not able to derive simple update formulas for the resources related to feasible delivery times of open requests. We suspect that if there are REFs for these resources carrying along several auxiliary resources needed for the calculations is necessary. It seems also mandatory for the computation of these resources to know the actual node sequence $\mathcal{P}_{\ell'}$ represented by label ℓ'.

In our algorithm, the earliest and latest delivery times are computed from scratch within each label. To do so, we use a generalization of the procedure of Tang *et al.* (2010) to obtain a feasible schedule with early-as-possible service times for all nodes which provides the values $ed_{\ell'}^i, i \in O_{\ell'}$. Starting from this schedule, we repeatedly delay the service at distinct nodes to obtain the remaining delivery times. This is done using an adapted version of the forward time slack originally introduced by Savelsbergh (1992) for the TSPTW and later generalized by Cordeau and Laporte (2003) for the DARP. Note that it is not necessary to consider the complete path $\mathcal{P}_{\ell'}$ for these computations. Instead, it is sufficient to take into account the subpath between the node at which the vehicle was empty for the last time and the current node $\eta_{\ell'}$.

The elimination of labels with no feasible completion to node $2n+1$ makes use of both the earliest possible delivery times ed_ℓ^i and the latest possible delivery times ld_ℓ^i. With this information, the label elimination strategy is very effective.

4. Branch-and-cut-and-price algorithm

This section briefly describes the main components of our basic branch-and-cut-and-price algorithm. Based on this algorithm, we devise four different integer column-generation approaches to the VRPTWT-SPD. Each of the approaches formulates the master problem on a different variable set implying subproblem SP, SP_{min}, SP^{max}, or SP^{max}_{min}. The subproblems are solved using the respective labeling algorithms of Sections 3.1 - 3.4.

Preprocessing. Time-window tightening and arc elimination is performed according to the rules proposed by Desrochers *et al.* (1992), Dumas *et al.* (1991), and Cordeau (2006) for the PDPTW or tailored to the PDPTW or the DARP. The integration of minimum and maximum ride-time constraints into these rules is straightforward. Note that all preprocessing steps have to be consistent with the dominance rule used to solve the subproblem. This means that one has to be careful when excluding arcs that cannot be part of a feasible VRPTWTSPD-route and at the same time using a formulation with a subproblem other than SP^{max}_{min}.

Assume, e.g., that it is infeasible in the presence of minimum and maximum ride times to have requests i and k on board of the vehicle at the same time. Thus, arc $(k, i+n)$ could be eliminated during preprocessing. Consider now an approach using SP as subproblem. Let ℓ_1 with $\mathcal{P}_{\ell_1} = (0, i, k)$ and ℓ_2 with $\mathcal{P}_{\ell_2} = (0, i, j, k)$ be two labels and suppose that ℓ_1 dominates ℓ_2 in the sense of *Dom-SP*. Although visiting node $i + n$ after node k always leads to an infeasible path in the VRPTWTSPD-sense, it may still be feasible for SP. Thus, it might be possible to feasibly complete ℓ_2 to path $\mathcal{P}_2 = (0, i, j, k, j + n, i + n, k + n, 2n + 1)$ while the corresponding completed path $\mathcal{P}_1 = (0, i, k, i + n, k + n, 2n + 1)$ associated with ℓ_1 cannot be generated because arc $(k, i + n)$ has been eliminated. As a result, dominance rule $Dom - SP$ is not valid anymore in this case.

Consequently, dominance rules other than $Dom - SP$, $Dom - SP^{max}$, and $Dom - SP_{min}$ have to be used when utilizing all information on VRPTWTSPD-feasibility of routes for preprocessing. The alternative is to exclude the information on VRPTWTSPD-feasibility during preprocessing that is also relaxed within the respective subproblem.

Pricing problem heuristics. To speed up the column-generation process, heuristics can be used to identify negative reduced-cost columns fast. When the heuristics are unable to find any more such columns, one has to resort to an exact method to solve the subproblem. In our algorithms, we use two straightforward pricing problem heuristics. One is to solve a more relaxed subproblem, e.g., solving SP when actually having to solve SP^{max}, and drop all routes that are infeasible for the actual subproblem. The other is to solve the subproblem on a reduced network only. Preliminary computational tests indicated that the benefits from using these heuristics were rather limited for all algorithms.

A different strategy related to pricing problem heuristics is the following: For the approaches based on the relaxed subproblems SP, SP_{min}, and SP^{max}, we perform full preprocessing, i.e., we include full information on VRPTWTSPD-feasibility during preprocessing, which means that the respective dominance rules $Dom-SP$, $Dom-SP_{min}$, and $Dom-SP^{max}$ are not valid for solving the resulting subproblems. However, algorithms with these dominance criteria can still be used as very effective pricing problem heuristics. Corresponding exact methods are obtained by using the weaker dominance rules in which the condition $O_{\ell_1} \subseteq O_{\ell_2}$ is replaced by $O_{\ell_1} = O_{\ell_2}$. In preliminary computational tests, this strategy has proven to be significantly superior to the one considering only preprocessing steps that are consistent with the stronger dominance rules $Dom - SP$, $Dom - SP_{min}$, and $Dom - SP^{max}$.

Cutting Planes. In our branch-and-cut-and-price algorithms, we use the following types of valid inequalities: 2-path inequalities (Kohl *et al.*, 1999), rounded capacity inequalities in a form proposed by Ropke and Cordeau (2009) for the PDPTW, fork inequalities (Ropke *et al.*, 2007), and two different liftings of IPEC introduced by Ascheuer *et al.* (2000) for the TSPTW and Cordeau (2006) for the DARP. Heuristic separation procedures proposed by Ropke and Cordeau (2009) are used to separate 2-path inequalities, rounded capacity inequalities, and fork inequalities. For the exact separation of the lifted IPEC we use a straightforward enumeration procedure (see Ascheuer *et al.*, 2000). VRPTWTSPD-feasibility of an integer solution obtained by approaches using a relaxed variable set Ω' is, thus, guaranteed by the lifted IPEC.

Branching strategy and node selection. A hierarchical branching scheme is used to obtain integer solutions in our algorithms. We first branch on the number of vehicles, if fractional. We then branch on the outflow of a node set of size two. Both branching rules are enforced by adding a single linear constraint to the master problem. The structure of the subproblems remains unchanged.

The branch-and-bound tree is explored with a best-first strategy and no upper bounds are given to the algorithm.

5. Computational results

This section summarizes the computational experiments that we have conducted to compare the performance of the four different branch-and-cut-and-price approaches to the VRPTWTSPD. All algorithms described in this paper were implemented in C++ using CPLEX 12.2 as LP-solver. Arc costs and travel times are computed with double precision. The experiments were performed on a standard PC with an Intel(R) Core(TM)2 Duo E8400 at 3.0 GHz with 4.0 GB main memory using a single thread only. The time limit was set to one hour.

The test instances used in the computational study are all based on the benchmark set for the DARP originally introduced by Cordeau (2006) and later extended by instances with larger problem sizes by Ropke *et al.* (2007). For a detailed description of these instances and their generation we refer to Cordeau (2006).

The DARP benchmark set consists of random Euclidean instances with problem sizes reaching from 2 vehicles and 16 customer requests to 8 vehicles and 96 customer requests. All instances are characterized by small vehicle capacities and narrow time windows. To also consider harder instances in which these constraints are less restrictive, three new instances were constructed from each original instance by enlarging both capacities and time-window lengths by factors 4/3, 5/3, and 6/3.

In the original benchmark set, there are two subsets of instances (type `a` and type `b`) with different characteristics regarding customer demand and vehicle capacity. Moreover, the maximum ride times are specified by a fixed number for each subset and are identical for all requests and all instances of the given subset. We did not use this fixed maximum ride time for our test instances. Instead, we modeled both ride times proportional to the direct travel time between pickup and delivery node of a request. More precisely, the maximum ride time of a request is equal to the product of the direct travel time and a random number chosen according to a uniform distribution over a given interval. To generate instances with different characteristics regarding the tightness of the ride-time constraints, we considered the two intervals $[2.25, 2.75]$ (for more restrictive maximum ride times) and $[2.75, 3.25]$ (for less restrictive maximum ride times). The minimum ride times were specified in an analog fashion using the intervals $[1.75, 2.25]$ and $[1.25, 1.75]$ for generating instances with more restrictive and less restrictive minimum ride times, respectively.

The complete benchmark comprises 672 instances labeled in the form `RT-TW-iK-n`, where `n` denotes the number of requests, `K` denotes the number of vehicles, and $\texttt{i} \in \{\texttt{a}, \texttt{b}\}$ denotes the subset the instance originates from. Moreover, `TW` = `A` refers to the original instances with small vehicle capacities and time-window lengths, while `TW` = `B`, `TW` = `C` and `TW` = `D` denote the instances in which these values have been enlarged by factor 4/3, 5/3, and 6/3, respectively. The characteristics regarding ride times are specified by $\texttt{RT} \in \{\texttt{MM}, \texttt{ML}, \texttt{LM}, \texttt{LL}\}$, where `M` and `L` indicate the more restrictive and less restrictive cases, respectively, while the first character refers to minimum ride times and the second character refers to maximum ride times. Note that some of the small instances are infeasible in the presence of minimum and maximum ride-time constraints. We allowed the use of additional vehicles to obtain well-defined instances in these cases. All instances are available at `http://logistik.bwl.uni-mainz.de/Dateien/vrptwtspd.zip`.

Table 3 summarizes our results averaged over all benchmark instances. Tables 4 and 5 present averaged results for the subclasses `A`, `B`, `C`, `D` and `MM`, `LM`, `ML`, `MM`, respectively. More detailed results can be found in Table 7 in Section B of the Appendix. They all report the following columns:

opt^{tree} number of optimal solutions obtained by respective algorithm

	opt^{tree}	$time$	opt^{root}	gap^{root}	$cuts$	$nodes$
IMP^{max}_{min}	653	204	345	0.12	8	19
$IMP\text{-}I^{max}$	652	194	298	0.17	42	43
$IMP\text{-}I_{min}$	552	781	296	0.29	193	35
$IMP\text{-}I$	540	852	269	0.33	214	35

Table 3: Summary results aggregated over all instances

		opt^{tree}	$time$	opt^{root}	gap^{root}	$cuts$	$nodes$
A (orig.)	IMP^{max}_{min}	166	122	111	0.06	5	13
	$IMP\text{-}I^{max}$	165	109	102	0.08	23	23
	$IMP\text{-}I_{min}$	163	168	107	0.09	70	14
	$IMP\text{-}I$	162	192	99	0.10	82	15
B (4/3)	IMP^{max}_{min}	166	109	92	0.08	6	12
	$IMP\text{-}I^{max}$	165	122	81	0.10	32	26
	$IMP\text{-}I_{min}$	154	401	77	0.15	137	23
	$IMP\text{-}I$	152	451	70	0.17	157	28
C (5/3)	IMP^{max}_{min}	164	193	83	0.13	8	21
	$IMP\text{-}I^{max}$	163	163	65	0.18	47	38
	$IMP\text{-}I_{min}$	135	903	63	0.31	224	41
	$IMP\text{-}I$	131	1034	54	0.35	248	46
D (6/3)	IMP^{max}_{min}	157	391	59	0.20	12	35
	$IMP\text{-}I^{max}$	159	381	50	0.30	68	93
	$IMP\text{-}I_{min}$	100	1653	49	0.61	342	58
	$IMP\text{-}I$	95	1730	46	0.68	371	57

Table 4: Aggregated results for subclasses A, B, C, and D

$time$ average computation time in seconds
opt^{root} number of instances solved to optimality in the root node
gap^{root} average percentage integrality gap in the root node
$cuts$ average number of cuts generated
$nodes$ average number of nodes solved

The results in Table 3 indicate that algorithms IMP^{max}_{min} and $IMP\text{-}I^{max}$ are clearly superior to $IMP\text{-}I_{min}$ and *IMP-I*. The overall performance of the two stronger approaches IMP^{max}_{min} and $IMP\text{-}I^{max}$ is comparable. In total, IMP^{max}_{min} is able to solve 653 out of the 672 instances to optimality, one more than $IMP\text{-}I^{max}$. Regarding computation times, $IMP\text{-}I^{max}$ is on average slightly faster than IMP^{max}_{min}. Algorithm $IMP\text{-}I_{min}$ is inferior to both of the former approaches regarding both computation times and number of solved instances. *IMP-I* performs even worse on both numbers.

The superiority of IMP^{max}_{min} and $IMP\text{-}I^{max}$ over $IMP\text{-}I_{min}$ and *IMP-I* can be attributed to the following reasons: First, the root node lower bounds of IMP^{max}_{min} and $IMP\text{-}I^{max}$ are significantly stronger resulting in smaller search trees for these approaches. Second, for $IMP\text{-}I_{min}$ and *IMP-I* substantially more cuts are added to the *LMP*s severely complicating their reoptimization. This is also the reason why the average number of solved nodes is smaller for approaches $IMP\text{-}I_{min}$ and *IMP-I* compared to approach $IMP\text{-}I^{max}$. While $IMP\text{-}I^{max}$ explores a huge number of nodes when solving difficult instances, algorithms $IMP\text{-}I_{min}$ and *IMP-I* spend a lot of time reoptimizing the *LMP*s and, thus, can solve only few nodes within the time limit. When comparing instances solved by all approaches, the number of explored nodes is indeed much higher for approaches $IMP\text{-}I_{min}$ and *IMP-I*.

The more disaggregated results in Tables 4 and 5 indicate that all findings from the overall results regarding the performance of the different approaches do also hold for all subclasses of instances. This means that the characteristics of the ride-time constraints have only limited influence on the relation of

		opt^{tree}	$time$	opt^{root}	gap^{root}	$cuts$	$nodes$
MM	IMP^{max}_{min}	167	67	103	0.08	7	11
	$IMP\text{-}I^{max}$	167	70	85	0.11	46	21
	$IMP\text{-}I_{min}$	148	584	91	0.15	185	18
	$IMP\text{-}I$	145	644	80	0.19	208	18
LM	IMP^{max}_{min}	166	127	92	0.08	9	19
	$IMP\text{-}I^{max}$	168	79	85	0.11	31	27
	$IMP\text{-}I_{min}$	143	688	79	0.22	207	18
	$IMP\text{-}I$	139	748	77	0.25	223	18
ML	IMP^{max}_{min}	161	249	72	0.15	7	19
	$IMP\text{-}I^{max}$	156	339	61	0.25	57	70
	$IMP\text{-}I_{min}$	131	913	61	0.38	192	49
	$IMP\text{-}I$	128	1012	53	0.45	221	49
LL	IMP^{max}_{min}	159	371	78	0.16	9	27
	$IMP\text{-}I^{max}$	161	287	67	0.19	35	55
	$IMP\text{-}I_{min}$	130	939	65	0.40	189	56
	$IMP\text{-}I$	128	1003	59	0.42	206	56

Table 5: Aggregated results for subclasses MM, LM, ML, and LL

the strengths of the considered algorithms. This is also true for vehicle capacity, customer demands, and time-window lengths.

Another interesting result of our experiments is that handling the maximum ride-time constraints in the subproblem seems to be more important than integrating the minimum ride times into the subproblem. Our interpretation is that the minimum ride-time constraints are often satisfied without explicitly considering them for the following reasons: When the time windows are narrow, many customer requests are picked up at their origin node i at time a_i. In these cases, the time-window tightening rules ensure that the minimum ride times are respected. With wide time windows, on the other hand, several other customer nodes are often visited in between the pickup and delivery of a request. This increases the ride times of the respective request compared to the direct travel times so that the minimum ride-time constraints might already be satisfied.

Table 6 reports detailed results for the stronger approaches IMP^{max}_{min} and $IMP\text{-}I^{max}$ on all 24 instances that are solved to optimality by at most one of the algorithms. Optimal solution values for all instances together with the computation times of the approaches IMP^{max}_{min} and $IMP\text{-}I^{max}$ can be found in Table 8-11 in Section B of the Appendix. In Table 6, the following notation is used:

inst	name of the instance
$opt/\overline{ub}$	value of the optimal solution or best known upper bound (overlined)
lb^{tree}	lower bound provided by the respective algorithm within the time limit, * if instance is solved to optimality within the time limit
gap^{tree}	percentage integrality gap when reaching the time limit
lb^{root}	root node lower bound
gap^{root}	percentage integrality gap in the root node
time	computation time in seconds, *1h* if unable to solve instance within time limit
cuts	number of cuts generated
nodes	number of nodes solved

In total, 15 instances were solved by neither of the algorithms within the time limit. We were able to close the integrality gap for ten of these instances using different algorithmic settings and/or allowing more computation time. For the remaining five instances, only an upper bound is reported.

		IMP^{max}_{min}							$IMP\text{-}I^{max}$						
inst	$opt/\overline{ub}$	lb^{tree}	gap^{tree}	lb^{root}	gap^{root}	time	cuts	nodes	lb^{tree}	gap^{tree}	lb^{root}	gap^{root}	time	cuts	nodes
ML-A-a6-72	908.54	*	0.00	904.84	0.41	1837	10	264	906.50	0.23	900.95	0.84	*1h*	191	681
ML-A-a8-96	1149.95	1149.26	0.06	1146.05	0.34	*1h*	32	214	1149.67	0.02	1145.16	0.42	*1h*	152	500
LL-A-a7-84	981.31	979.87	0.15	974.14	0.74	*1h*	15	348	980.70	0.06	972.07	0.95	*1h*	167	634
ML-B-a8-96	1114.48	*	0.00	1112.50	0.18	938	18	27	1113.08	0.13	1108.86	0.51	*1h*	186	345
LL-B-a7-84	952.52	951.16	0.14	946.37	0.65	*1h*	47	264	951.45	0.11	945.13	0.78	*1h*	178	563
LL-B-b8-96	$\overline{1107.70}$	1106.38	0.12	1102.17	0.50	*1h*	14	288	1106.17	0.14	1101.27	0.58	*1h*	72	635
MM-C-b8-80	943.70	943.50	0.02	939.65	0.43	*1h*	19	522	942.99	0.08	938.64	0.54	*1h*	204	626
ML-C-a8-96	$\overline{1101.72}$	1095.80	0.54	1093.28	0.77	*1h*	25	70	1093.84	0.72	1089.58	1.11	*1h*	223	316
ML-C-b8-80	925.53	*	0.00	921.44	0.44	1000	14	134	925.50	0.00	920.56	0.54	*1h*	187	883
ML-C-b8-96	1047.67	1046.33	0.13	1043.78	0.37	*1h*	1	77	1046.61	0.10	1042.98	0.45	*1h*	163	362
LL-C-a8-96	1087.91	1085.35	0.24	1083.48	0.41	*1h*	11	42	1087.46	0.04	1083.14	0.44	*1h*	134	291
LM-D-a7-84	953.92	952.88	0.11	948.74	0.55	*1h*	54	292	*	0.00	947.89	0.64	2681	162	476
LM-D-b8-64	695.72	695.36	0.05	689.27	0.94	*1h*	20	842	*	0.00	688.65	1.03	3406	197	1365
ML-D-a6-72	853.62	852.59	0.12	847.08	0.77	*1h*	33	222	852.57	0.12	845.67	0.94	*1h*	208	573
ML-D-a7-84	$\overline{927.15}$	925.07	0.22	921.71	0.59	*1h*	34	78	923.07	0.44	917.60	1.04	*1h*	220	424
ML-D-a8-96	$\overline{1070.58}$	1068.11	0.23	1066.54	0.38	*1h*	9	17	1068.11	0.23	1064.06	0.61	*1h*	118	283
ML-D-b7-84	1016.84	1015.89	0.09	1009.86	0.69	*1h*	12	101	1014.04	0.28	1005.67	1.11	*1h*	211	403
ML-D-b8-80	869.68	*	0.00	866.71	0.34	1690	40	114	868.50	0.14	862.14	0.88	*1h*	287	680
ML-D-b8-96	1021.89	*	0.00	1021.08	0.08	486	3	10	1020.47	0.14	1015.45	0.63	*1h*	194	303
LL-D-a7-84	909.82	907.67	0.24	904.34	0.61	*1h*	47	81	907.81	0.22	902.27	0.84	*1h*	134	369
LL-D-a8-96	1057.65	1056.94	0.07	1055.32	0.22	*1h*	12	11	1057.65	0.00	1054.36	0.31	*1h*	59	155
LL-D-b6-60	726.76	725.11	0.23	719.90	0.95	*1h*	4	523	*	0.00	719.24	1.05	2497	133	893
LL-D-b7-84	1012.45	1011.80	0.06	1008.15	0.43	*1h*	23	132	*	0.00	1007.48	0.49	2776	144	292
LL-D-b8-80	$\overline{861.25}$	858.88	0.28	854.37	0.81	*1h*	20	148	857.08	0.49	849.67	1.36	*1h*	227	710
∅			0.13		0.52		22	201		0.15		0.75		173	532

Table 6: Results for IMP^{max}_{min} and $IMP\text{-}I^{max}$ on instances solved by at most one algorithm

6. Conclusion

In this paper, we introduced the Vehicle Routing Problem with Time Windows and Temporal Synchronized Pickup and Delivery (VRPTWTSPD) as the prototypical VRP with temporal intra-route synchronization. In the VRPTWTSPD, vehicle routes have to satisfy pairing and precedence, capacities, and time windows. Additionally, temporal synchronization constraints couple the service times at the pickup and delivery locations of the customer requests in the following way: A delivery node has to be serviced within a given minimum and maximum time lag (called ride time) after the service at the corresponding pickup node has been completed.

The minimum and maximum ride-time constraints severely complicate the subproblem of the natural column-generation formulation of the VRPTWTSPD and it is not clear if their integration into the subproblem pays off in an integer column-generation approach. We, therefore, developed four solution approaches to the VRPTWTSPD based on column-generation formulations with differing subproblems. Two of these subproblems, the natural subproblem of the VRPTWTSPD that integrates all constraints relating to single routes and the one in which the maximum ride-time constraints are relaxed, were considered for the first time in this paper. New dominance rules and labeling algorithms for their solution have been derived. Extensive computational experiments demonstrated the applicability of these labeling algorithms in the sense that they are capable of solving subproblems arising in state-of-the-art benchmark instances in reasonable time.

The computational results also indicated a clear ranking of the four presented algorithms for solving the VRPTWTSPD. The strongest approaches are the approach based on the natural column-generation formulation and the one that handles only the maximum ride times in the subproblem. They performed comparably well and were consistently significantly stronger than the approach with only the minimum ride times in the subproblem. This was in turn slightly stronger than the approach in which both ride-time constraints are relaxed in the subproblem. We conclude that the integration of temporal intra-route synchronization constraints into the column-generation subproblem is beneficial and that it is particularly rewarding for the maximum ride-times.

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Appendix

A. Proofs

For the presentation of the proofs some additional notation is necessary. For any set of numbers M and any number n, let $n + M = \{n + m : m \in M\}$. Moreover, given a sequence $\mathcal{P} = (h_1, ..., h_q)$ (path or schedule) and a set of numbers M, let $\mathcal{P} \setminus M$ be the sub-sequence of $\mathcal{P}$ where h_i is removed if $h_i = m$ for all $m \in M$.

We first sketch the proof of Proposition 1 (see Dumas *et al.*, 1991) which is the basis for Propositions 3 and 4.

Proof of Proposition 1: The basic idea of the proof is as follows. Let Q_2 be a feasible completion to ℓ_2, i.e., Q_2 is a extension of $\mathcal{P}_{\ell_2}$ to node $2n+1$ such that the path $\mathcal{P}_2 = (\mathcal{P}_{\ell_2}, Q_2)$ is feasible. Consider the path $\mathcal{P}_1 = (\mathcal{P}_{\ell_1}, Q_1)$ where $Q_1 = Q_2 \setminus \{n + (O_{\ell_2} \setminus O_{\ell_1})\}$ is the completion to ℓ_1 resulting from Q_2 by skipping the delivery nodes of all additional open requests of ℓ_2. Clearly, pairing and precedence constraints are then satisfied by $\mathcal{P}_1$. Using that $\mathcal{P}_2$ is feasible, the relations in Proposition 1 also ensure that $\mathcal{P}_1$ is feasible with respect to capacity and time windows. Furthermore, when the DTI holds the cost of $\mathcal{P}_1$ is always better than that of $\mathcal{P}_2$. □

Proof of Lemma 1: $\underline{T}_{\mathcal{P}}$ is feasible if it satisfies inequalities (1)–(3). By definition of $\underline{\tau}_i$, there exists for each $i = 1, ..., q$ a feasible schedule $T^i_{\mathcal{P}} = (\tau^i_1, ..., \underline{\tau}_i, ..., \tau^i_q)$ with $\tau^i_j \geq \underline{\tau}_j$ for all $j \neq i$. It follows immediately that $\underline{T}_{\mathcal{P}}$ satisfies (2) for all $i = 1, ..., q$. Using $\tau^{i+1}_i + t_{h_i h_{i+1}} \leq \underline{\tau}_{i+1}$ and $\underline{\tau}_i \leq \tau^{i+1}_i$ it follows that $\underline{T}_{\mathcal{P}}$ satisfies (1) for all $i = 1, ..., q-1$. Also, using $\tau^j_i + \underline{L}_{h_i} \leq \underline{\tau}_j$ and $\underline{\tau}_i \leq \tau^j_i$ it follows that $\underline{T}_{\mathcal{P}}$ satisfies (3) for all $h_i = h_j - n$. □

Proof of Proposition 3: The proof is similar to the proof of Proposition 1. Following the same argumentation and using the same notation, it remains to show that $\mathcal{P}_1$ also respects minimum ride-time constraints, i.e., we have to show that there exists a feasible time schedule $T_{\mathcal{P}_1}$ for $\mathcal{P}_1$.

Let $T_{\mathcal{P}_2} = (T_{\mathcal{P}_{\ell_2}}, T_{Q_2})$ be a feasible schedule for $\mathcal{P}_2$ with $T_{Q_2} = (\tau_{q+1}, ..., \tau_r)$. Denote by $\underline{T}_{\mathcal{P}_{\ell_1}} = (\underline{\tau}_1, ..., \underline{\tau}_q) \in \mathscr{T}_{\mathcal{P}_{\ell_1}}$ the time schedule for $\mathcal{P}_{\ell_1}$ that minimizes the start of service at all nodes and by $T_{Q_1} = T_{Q_2} \setminus \{\tau_i : h_i - n \in O_{\ell_2} \setminus O_{\ell_1}\}$ the schedule for Q_1 that assigns each node h_i of Q_1 the same start of service τ_i as in T_{Q_2}. Then, using that $\underline{T}_{\mathcal{P}_{\ell_1}}$ and $T_{\mathcal{P}_2}$ are feasible, $\underline{\tau}_q = t_{\ell_1} \leq t_{\ell_2}$, and $ed^{h_i}_{\ell_1} \leq ed^{h_i}_{\ell_2} \leq \tau_i$ for all nodes h_i of Q_1 with $h_i - n \in O_{\ell_1}$ it follows that the schedule $T_{\mathcal{P}_1} = (\underline{T}_{\mathcal{P}_{\ell_1}}, T_{Q_1})$ is feasible. □

Proof of Proposition 4: The basic course of the proof is similar to that in the proof of Proposition 3. With the same argumentation and notation, it again remains to show that there exists a feasible time schedule $T_{\mathcal{P}_1}$ for $\mathcal{P}_1$.

Let $\tau^{\ell_2}_q$ be the start of service at the current node $h_q = \eta_{\ell_1}$ within the schedule $T_{\mathcal{P}_2}$. Denote by $\overline{\tau}'_i(\tau^{\ell_2}_q) = \max_{T_{\mathcal{P}_{\ell_1}} \in \mathscr{T}_{\mathcal{P}_{\ell_1}}(\tau^{\ell_2}_q), \tau_j = \tau'_j \, \forall h_j \in O_{\ell_1} \setminus \{h_i\}} \{\tau_i\}$ the latest feasible start of service at node $h_i \in O_{\ell_1}$ given a feasible choice τ'_j for the values $\tau_j, h_j \in O_{\ell_1} \setminus \{h_i\}$, i.e., a choice such that a feasible schedule $T_{\mathcal{P}_{\ell_1}} = (\tau_1, ..., \tau_q) \in \mathscr{T}_{\mathcal{P}_{\ell_1}}(\tau^{\ell_2}_q)$ with $\tau_j = \tau'_j$ for all $h_j \in O_{\ell_1} \setminus \{h_i\}$ exists. We first show that $\overline{\tau}'_i(\tau^{\ell_2}_q) \geq \overline{\tau}^{O_{\ell_1}}_i(\tau^{\ell_2}_q)$ holds.

The proof is constructive. Denote by $\overline{\tau}'_k(\tau^{\ell_2}_q)$ and $\overline{\tau}^{O_{\ell_1}}_k(\tau^{\ell_2}_q)$ also the latest start of service at all other nodes $h_k \neq h_i$ subject to the choices τ'_j and $\underline{\tau}_j$ for all $h_j \in O_{\ell_1} \setminus \{h_i\}$, respectively. Moreover for ease of notation, the explicit functional dependence on the start of service at the current node is omitted in the following, as all considered values relate to schedules with $\tau_q \leq \tau^{\ell_2}_q$. We start from the schedule $\overline{T}'_{\mathcal{P}_{\ell_1}} = (\overline{\tau}'_1, ..., \overline{\tau}'_q) \in \mathscr{T}_{\mathcal{P}_{\ell_1}}$. Feasibility of $\overline{T}'_{\mathcal{P}_{\ell_1}}$ follows analog to the proof of Lemma 1. Thus, $\overline{T}'_{\mathcal{P}_{\ell_1}}$ satisfies the constraint system (1)–(4), $\tau_q \leq \tau^{\ell_2}_q$, and $\tau_j = \tau'_j, h_j \in O_{\ell_1} \setminus \{h_i\}$ of the maximization problems of $\overline{\tau}'_k, k = 1, ..., q$. Replacing equalities $\tau_j = \tau'_j$ by $\tau_j = \underline{\tau}_j$ for all $h_j \in O_{\ell_1} \setminus \{h_i\}$ leads to the constraints system for the maximization problems of $\overline{\tau}^{O_{\ell_1}}_i, k = 1, ..., q$. The modified equalities directly impose stronger bounds on all values τ_{j-1} because of inequalities (1). The case $h_{j-1} \in O_{\ell_1} \setminus \{h_i\}$ can be neglected, as the value of τ_{j-1} is then fixed to $\underline{\tau}_{j-1}$ and consistency of $\underline{\tau}_{j-1}$ and $\underline{\tau}_j$ follows directly from feasibility of the choice $\underline{\tau}_k, h_k \in O_{\ell_1}$ which can again be shown analog to the proof of Lemma 1.

In the case $h_{j-1} \notin O_{\ell_1} \setminus \{h_i\}$, the maximal value at position $j-1$ decreases from $\overline{\tau}'_{j-1}$ to $\overline{\tau}^{O_{\ell_1}}_{j-1} = \min\{\overline{\tau}'_{j-1}, \underline{\tau}_j - t_{h_{j-1}h_j}\} = \overline{\tau}'_{j-1} + \Delta_{j-1}$ with $\Delta_{j-1} \leq 0$. A reduced value $\overline{\tau}^{O_{\ell_1}}_{j-1}$ compared to $\overline{\tau}'_{j-1}$ may in turn influence the maximal values at positions $j-2$, k with $h_k = h_{j-1} - n$, and l with $h_l = h_{j-1} + n$ because of inequalities (1), (3), and (4), respectively. W.l.o.g. consider the impact on the maximal value at position $j-2$ which decreases from $\overline{\tau}'_{j-2}$ to

$$\begin{aligned}
\overline{\tau}^{O_{\ell_1}}_{j-2} &= \min\{\overline{\tau}'_{j-2}, \overline{\tau}^{O_{\ell_1}}_{j-1} - t_{h_{j-2}h_{j-1}}\} \\
&= \overline{\tau}'_{j-2} + \min\{0, \overline{\tau}^{O_{\ell_1}}_{j-1} - t_{h_{j-2}h_{j-1}} - \overline{\tau}'_{j-2}\} \\
&= \overline{\tau}'_{j-2} + \min\{0, \overline{\tau}'_{j-1} + \Delta_{j-1} - t_{h_{j-2}h_{j-1}} - \overline{\tau}'_{j-2}\} \\
&= \overline{\tau}'_{j-2} + \Delta_{j-2},
\end{aligned}$$

with $\Delta_{j-2} \leq 0$. As $\overline{\tau}'_{j-1} - t_{h_{j-2}h_{j-1}} - \overline{\tau}'_{j-2} \geq 0$ it follows that $0 \leq \Delta_{j-2} \leq \Delta_{j-1}$. Thus, the value by which $\overline{\tau}^{O_{\ell_1}}_{j-2}$ is decreased compared to $\overline{\tau}'_{j-2}$ is strictly not larger than the value by which $\overline{\tau}^{O_{\ell_1}}_{j-1}$ is decreased compared to $\overline{\tau}'_{j-1}$. The same result is obtained for the maximal values at positions k with $h_k = h_{j-1} - n$ and l with $h_l = h_{j-1} + n$.

Decreased values $\overline{\tau}^{O_{\ell_1}}_{j-2}$, $\overline{\tau}^{O_{\ell_1}}_{k}$ with $h_k = h_{j-1} - n$, and $\overline{\tau}^{O_{\ell_1}}_{l}$ with $h_l = h_{j-1} + n$ in turn constrain the maximal values at several other positions m within the path in an analog fashion as $\overline{\tau}^{O_{\ell_1}}_{j-1}$ constrained themselves as shown above. As a result, we get $\overline{\tau}^{O_{\ell_1}}_{m} = \overline{\tau}'_m + \Delta_m$ with $\Delta_m \leq 0$. Moreover, $0 \leq \Delta_m \leq \Delta_{j-2}, \Delta_k, \Delta_l$ also holds, i.e., again the decreasing effects on values $\overline{\tau}^{O_{\ell_1}}_{m}$ compared to $\overline{\tau}'_m$ are strictly not larger than the decreasing effect on $\overline{\tau}^{O_{\ell_1}}_{j-2}$, $\overline{\tau}^{O_{\ell_1}}_{k}$, and $\overline{\tau}^{O_{\ell_1}}_{l}$ compared to $\overline{\tau}'_{j-2}$, $\overline{\tau}'_k$, and $\overline{\tau}'_l$, respectively.

Iteratively propagating the stronger conditions resulting from decreased values $\overline{\tau}^{O_{\ell_1}}_{k}$, $k = 1, ..., q$ compared to $\overline{\tau}'_k$ in the constraint system allows to construct $\overline{T}^{O_{\ell_1}} = (\overline{\tau}^{O_{\ell_1}}_{1}, ..., \overline{\tau}^{O_{\ell_1}}_{q})$ with $\overline{\tau}^{O_{\ell_1}}_{k} = \overline{\tau}'_k + \Delta_k$ and $\Delta_k \leq 0$ for all $k = 1, ..., q$. If there are more than just one decreasing effects that have not been propagated yet, they are processed in non-increasing absolute values. Using the property that the decreasing effects Δ_k are non-increasing from propagation step to propagation step, there exists a unique value by which the maximal start of service at each position is decreased and no cycle effects can occur. As a result, $\overline{\tau}'_i(\tau^{\ell_2}_q) \geq \overline{\tau}^{O_{\ell_1}}_{i}(\tau^{\ell_2}_q)$ holds for any feasible choice τ'_j of the values $\tau_j, h_j \in O_{\ell_1} \setminus \{h_i\}$.

In an analog fashion, we can show that $\underline{\tau}'_i(\tau^{\ell_2}_q) \leq \underline{\tau}^{\overline{O}_{\ell_1}}_{i}(\tau^{\ell_2}_q)$ holds. Clearly, we also have that $\underline{\tau}'_i(\tau^{\ell_2}_q) \leq \overline{\tau}'_i(\tau^{\ell_2}_q)$. Moreover, it is easy to see that any convex combination of two feasible schedules is also a feasible schedule. Thus, given any feasible choice τ'_j for the values $\tau_j, h_j \in O_{\ell_1} \setminus \{h_i\}$ there exists for each $\tau^*_i \in [\underline{\tau}'_i(\tau^{\ell_2}_q), \overline{\tau}'_i(\tau^{\ell_2}_q)]$ with $\underline{\tau}'_i(\tau^{\ell_2}_q) \leq \underline{\tau}^{\overline{O}_{\ell_1}}_{i}(\tau^{\ell_2}_q)$ and $\overline{\tau}'_i(\tau^{\ell_2}_q) \geq \overline{\tau}^{O_{\ell_1}}_{i}(\tau^{\ell_2}_q)$ a feasible schedule with start of service τ^*_i at node $h_i \in O_{\ell_1}$.

Using this property, we can construct a feasible schedule $T_{\mathcal{P}_1} = (T^*_{\mathcal{P}_{\ell_1}}, T_{Q_1})$ as follows. As defined in the proof of Proposition 3, fix $T_{Q_1} = T_{Q_2} \setminus \{\tau_i : h_i - n \in O_{\ell_2} \setminus O_{\ell_1}\}$ to the schedule for Q_1 that assigns each node h_i of Q_1 the same start of service τ_i as in T_{Q_2}. Let $T_{\mathcal{P}_{\ell_1}} = (\tau_1, ..., \tau_q) \in \mathscr{T}_{\mathcal{P}_{\ell_1}}(\tau^{\ell_2}_q)$. For $T^*_{\mathcal{P}_{\ell_1}} = T_{\mathcal{P}_{\ell_1}}$, schedule $T_{\mathcal{P}_1}$ clearly satisfies inequalities (1) and (2). Also, it satisfies inequalities (3) and (4) for each request $h_i \notin O_{\ell_1}$.

If $T_{\mathcal{P}_1}$ also respects inequalities (3) and (4) for each request $h_i \in O_{\ell_1}$, $T_{\mathcal{P}_1}$ is feasible and the proof is complete. Otherwise, consider any request h_i for which either constraint (3) or (4) is violated. Denote by τ_k the start of service at the delivery node $h_k = h_i + n$ as fixed in schedule T_{Q_1}. To satisfy inequalities (3) and (4), $\tau_i \leq \tau_k - \underline{L}_{h_i}$ and $\tau_i \geq \tau_k - \overline{L}_{h_i}$ must hold, respectively, for the start of service τ_i at pickup node h_i within schedule $T^*_{\mathcal{P}_{\ell_1}}$. Let $\underline{\tau}'_i(\tau^{\ell_2}_q)$ and $\overline{\tau}'_i(\tau^{\ell_2}_q)$ be the minimal and maximal starts of service at node h_i, respectively, given the values $\tau_j, h_j \in O_{\ell_1} \setminus \{h_i\}$ as fixed within schedule $T_{\mathcal{P}_{\ell_1}}$. Then, $\tau_k - \underline{L}_{h_i} \geq ed^{h_i}_{\ell_2} - \underline{L}_{h_i} \geq$

$\overline{ed}^{h_i}_{\ell_1}(\tau_q^{\ell_2}) - \underline{L}_{h_i} \geq \underline{\tau}_i^{\overline{O}_{\ell_1}}(\tau_q^{\ell_2}) \geq \underline{\tau}'_i(\tau_q^{\ell_2})$ and $\tau_k - \overline{L}_{h_i} \leq ld^{h_i}_{\ell_2}(\tau_q^{\ell_2}) - \overline{L}_{h_i} \leq \underline{ld}^{h_i}_{\ell_1}(\tau_q^{\ell_2}) - \overline{L}_{h_i} \leq \overline{\tau}_i^{\underline{O}_{\ell_1}}(\tau_q^{\ell_2}) \leq \overline{\tau}'_i(\tau_q^{\ell_2})$ hold. Using the property shown above, there exists a feasible schedule $T'_{\mathcal{P}_{\ell_1}} = (\tau'_1, ..., \tau'_q)$ with $\tau'_j = \tau_j$ for all $h_j \in O_{\ell_1} \setminus \{h_i\}$ and any $\tau'_i = \tau^*_i \in \left[\underline{\tau}'_i(\tau_q^{\ell_2}), \overline{\tau}'_i(\tau_q^{\ell_2})\right]$. Thus for $T^*_{\mathcal{P}_{\ell_1}} = T'_{\mathcal{P}_{\ell_1}}$, $T_{\mathcal{P}_1}$ satisfies constraints (3) and (4) for request h_i. Moreover, all other constraints that have been respected in the case $T^*_{\mathcal{P}_{\ell_1}} = T_{\mathcal{P}_{\ell_1}}$ are still respected.

If for $T_{\mathcal{P}_1}$ inequalities (3) and (4) are still violated for some requests, we can iteratively repeat the same procedure just described. In each iteration one constraint that was previously violated gets satisfied. Thus, a feasible schedule $T_{\mathcal{P}_1}$ for path $\mathcal{P}_1$ eventually results which completes the proof. $\square$

Proof of Lemma 2: Note first that $\mathcal{X}(t) \notin \varnothing$ for all $t \geq t^*$. Also note that $\overline{X} = (\overline{x}_1, ..., \overline{x}_q) \in \mathcal{X}$, $\underline{X} = (\underline{x}_1, ..., \underline{x}_q) \in \mathcal{X}$, $\overline{X}(t) = (\overline{x}_1(t), ..., \overline{x}_q(t)) \in \mathcal{X}(t)$, $\underline{X}(t) = (\underline{x}_1(t), ..., \underline{x}_q(t)) \in \mathcal{X}(t)$, $\overline{X}^{\underline{S}}(t) = (\overline{x}_1^{\underline{S}}(t), ..., \overline{x}_q^{\underline{S}}(t)) \in \mathcal{X}^{\underline{S}}(t)$, and $\underline{X}^{\overline{S}}(t) = (\underline{x}_1^{\overline{S}}(t), ..., \underline{x}_q^{\overline{S}}(t)) \in \mathcal{X}^{\overline{S}}(t)$. The proofs are analog to the proof of Lemma 1.

1. The proof is constructive. We start from $\overline{X} \in \mathcal{X}$, i.e., the q-tuple with maximal values for all $\overline{x}_i, i = 1, ..., q$ satisfying (30)–(32). When considering $\mathcal{X}(t), t \geq t^*$ instead of $\mathcal{X}$, we have the additional constraint $x_q \leq t$. The case $\overline{x}_q \leq t$ is trivial. If $\overline{x}_q > t$ for some $t \geq t^*$, the maximal value at position q decreases to $\overline{x}_q(t) = \min\{\overline{x}_q, t\} = \overline{x}_q + \min\{0, t - \overline{x}_q\} = \overline{x}_q + \Delta_q(t)$ with $\Delta_q(t) = \min\{k_q^1, k_q^2 + t\} \leq 0$. If inequalities (31) are satisfied for the decreased value $\overline{x}_q(t)$ and all $\overline{x}_i, i < q$, then clearly $\overline{x}_j(t) = \overline{x}_j = \min\{k_i^1, k_i^2 + t\}$ for all $j = 1, ..., q$ with constants k_i^1 and k_i^2.

   If $\overline{x}_i + c_{iq} > \overline{x}_q(t)$ holds for some $i < q$ and $t \geq t^*$, then $\overline{x}_i(t) = \min\{\overline{x}_i, \overline{x}_q(t) - c_{iq}\}$ which can be rewritten in the forms

$$\begin{aligned}
\overline{x}_i(t) &= \min\{\overline{x}_i, \overline{x}_q(t) - c_{iq}\} \\
&= \min\{k_i^1, k_i^2 + t\} \\
&= \overline{x}_i + \min\{0, \overline{x}_q(t) - c_{iq} - \overline{x}_i\} \\
&= \overline{x}_i + \min\{0, \overline{x}_q - c_{iq} - \overline{x}_i + \Delta_q(t)\} \\
&= \overline{x}_i + \Delta_i(t),
\end{aligned}$$

   where $\Delta_i(t) = \min\{\tilde{k}_i^1, \tilde{k}_i^2 + t\} \leq 0$ with constants $\tilde{k}_i^1$ and $\tilde{k}_i^2$. As $\overline{x}_q - c_{iq} - \overline{x}_i \geq 0$, it follows that $0 \leq \Delta_i(t) \leq \Delta_q(t)$. Thus, the value by which $\overline{x}_i(t)$ is decreased compared to $\overline{x}_i$ is strictly not larger than the value by which $\overline{x}_q(t)$ is decreased compared to $\overline{x}_q$.

   Decreased values $\overline{x}_i(t)$ in turn impose stronger bounds compared to $\overline{x}_i$ on the maximal values at all positions $j < i$ and $k > i$ because of inequalities (31) and (32), respectively. Thus, the decreasing effects propagate through the constraint system in the same fashion as in the proof of Proposition 4. A schedule $\overline{X}(t) = (\overline{x}_1(t), ..., \overline{x}_q(t)) \in \mathcal{X}(t)$ with $\overline{x}_i(t) = \max\{k_j^1, k_j^2 + t\}\}$ for all $i = 1, ..., q, t \geq t^*$ can then be constructed analog to there.

2. It is straightforward to verify that $\underline{X} \in \mathcal{X}(t)$ and $\mathcal{X}(t) \subseteq \mathcal{X}$ for all $t \geq t^*$. Thus, $\underline{x}_i(t) = \underline{x}_i$ for all $i = 1, ..., q, t \geq t^*$.

3. $\overline{x}_i^{\underline{S}}(t)$ is given by $\max\{x_i\}$, s.t. (30)–(32), $x_q \leq t$, and $x_j \leq \underline{x}_j, j \in \underline{S}$. Obviously, the latter maximization problem is of the same structure as the one for $\overline{x}_i(t)$ and, thus, $\overline{x}_i^{\underline{S}}(t) = \max\{k_i^1, k_i^2 + t\}$ with constants k_i^1 and k_i^2 for all $i = 1, ..., q, t \geq t^*$.

4. Comparing the minimization problems for $\underline{x}_i(t)$ and $\underline{x}_i^{\overline{S}}(t)$, there are additional constraints $x_j \geq \overline{x}_j(t), j \in \overline{S}$ in the latter. When constructing $\underline{X}^{\overline{S}}(t) = (\underline{x}_1^{\overline{S}}(t), ..., \underline{x}_q^{\overline{S}}(t)) \in \mathcal{X}^{\overline{S}}(t)$ starting from $\underline{X}(t) =$

$(\underline{x}_1, ..., \underline{x}_q) \in \mathcal{X}(t)$, these constraints may force $\underline{x}_j^{\overline{S}}(t)$ to be increased compared to $\underline{x}_j$ for $j \in \overline{S}$ and we have

$$\underline{x}_j^{\overline{S}}(t) = \max\{\underline{x}_j, \overline{x}_j(t)\} \tag{36}$$
$$= \max\{\underline{x}_j, \min\{k_j^1, k_j^2 + t\}\} \tag{37}$$
$$= \underline{x}_j + \max\{0, \min\{k_j^1, k_j^2 + t\} - \underline{x}_j\} \tag{38}$$
$$= \underline{x}_j + \Delta_j(t), \tag{39}$$

with $\Delta_j(t) = \max\{\tilde{k}_j^1, \min\{\tilde{k}_j^2, \tilde{k}_j^3 + t\}\} \geq 0$. Increased values $\underline{x}_j^{\overline{S}}(t)$ compared to $\underline{x}_j$ might in turn influence other values and the increasing effects propagate throughout the constraint system.

A schedule $\underline{X}^{\overline{S}}(t) = (\underline{x}_1^{\overline{S}}(t), ..., \underline{x}_q^{\overline{S}}(t)) \in \mathcal{X}^{\overline{S}}(t)$ with $\underline{x}_i^{\overline{S}}(t) = \max\{k_j^1, \min\{k_j^2, k_j^3 + t\}\}$ for all $i = 1, ..., q, t \geq t^*$ can then be constructed in the same fashion as $\overline{X}(t)$ was constructed in the proof of the first property.

□

Proof of Proposition 5: We need to show that (34) implies $\overline{ed}_{\ell_1}^i(t) \leq ed_{\ell_2}^i, t \in [t_{\ell_2}, \tilde{b}_{\ell_2}]$ and that (35) implies $\underline{ld}_{\ell_1}^i(t) \geq ld_{\ell_2}^i(t), t \in [t_{\ell_2}, \tilde{b}_{\ell_2}]$. The latter follows directly from Proposition 5 of Gschwind and Irnich (2013). For the former, recall that $\overline{ed}_{\ell_1}^i(t) = \max\{k_i^1, \min\{k_i^2, k_i^3 + t\}\}$ and note that $\overline{ed}_{\ell_1}^i(t)$ is clearly non-decreasing in t.

Consider first the case $\overline{B}_{\ell_1}^i \leq \tilde{b}_{\ell_2}$. Using that $\overline{ed}_{\ell_1}^i(t)$ is constant for all $t \geq \overline{B}_{\ell_1}^i$, we have that $\overline{ed}_{\ell_1}^i(t) = \overline{ed}_{\ell_1}^i(\overline{B}_{\ell_1}^i) \leq ed_{\ell_2}^i$ for all $\overline{B}_{\ell_1}^i \leq t \leq \tilde{b}_{\ell_2}$. For all $t_{\ell_2} \leq t \leq \overline{B}_{\ell_1}^i$, clearly $\overline{ed}_{\ell_1}^i(t) \leq \overline{ed}_{\ell_1}^i(\overline{B}_{\ell_1}^i) \leq ed_{\ell_2}^i$ holds.

Consider now the case $\overline{B}_{\ell_1}^i > \tilde{b}_{\ell_2}$. If $\overline{ed}_{\ell_1}^i(\tilde{b}_{\ell_2}) \leq \overline{ed}_{\ell_1}^i(\overline{B}_{\ell_1}^i) - (\overline{B}_{\ell_1}^i - \tilde{b}_{\ell_2})$, then $\overline{ed}_{\ell_1}^i(t) \leq \overline{ed}_{\ell_1}^i(\tilde{b}_{\ell_2}) \leq \overline{ed}_{\ell_1}^i(\overline{B}_{\ell_1}^i) - (\overline{B}_{\ell_1}^i - \tilde{b}_{\ell_2}) \leq ed_{\ell_2}^i$ holds for all $t_{\ell_2} \leq t \leq \tilde{b}_{\ell_2}$. If $\overline{ed}_{\ell_1}^i(\tilde{b}_{\ell_2}) > \overline{ed}_{\ell_1}^i(\overline{B}_{\ell_1}^i) - (\overline{B}_{\ell_1}^i - \tilde{b}_{\ell_2})$ we differentiate two cases. First, if $\overline{B}_{\ell_1}^i = t_{\ell_1}$ it follows directly that $\overline{ed}_{\ell_1}^i(t) = \overline{ed}_{\ell_1}^i(\overline{B}_{\ell_1}^i) = \overline{ed}_{\ell_1}^i(t_{\ell_1}) \leq ed_{\ell_2}^i$ for all $t_{\ell_2} \leq t \leq \tilde{b}_{\ell_2}$. Second, if $\overline{B}_{\ell_1}^i > t_{\ell_1}$ we have that $\overline{ed}_{\ell_1}^i(t) < \overline{ed}_{\ell_1}^i(\overline{B}_{\ell_1}^i)$ for all $t < \overline{B}_{\ell_1}^i$ and consequently $\overline{ed}_{\ell_1}^i(\overline{B}_{\ell_1}^i) = k_i^2 = k_i^3 + \overline{B}_{\ell_1}^i$ must hold. Using $\overline{ed}_{\ell_1}^i(\tilde{b}_{\ell_2}) > \overline{ed}_{\ell_1}^i(\overline{B}_{\ell_1}^i) - (\overline{B}_{\ell_1}^i - \tilde{b}_{\ell_2})$ it follows that $\overline{ed}_{\ell_1}^i(\tilde{b}_{\ell_2}) > k_i^3 + \tilde{b}_{\ell_2}$ and, thus, $\overline{ed}_{\ell_1}^i(\tilde{b}_{\ell_2}) = k_i^1$. As a result, we have $\overline{ed}_{\ell_1}^i(t) = \overline{ed}_{\ell_1}^i(\tilde{b}_{\ell_2}) = k_i^1 = \overline{ed}_{\ell_1}^i(t_{\ell_1}) \leq ed_{\ell_2}^i$ for all $t_{\ell_2} \leq t \leq \tilde{b}_{\ell_2}$. □

B. Detailed computational results

Table 7 shows aggregated results for all algorithms and subclasses. It reports the following columns:

opt^{tree}	number of optimal solutions obtained by respective algorithm
time	average computation time in seconds
opt^{root}	number of optimal solutions obtained in the root node
gap^{root}	average percentage integrality gap in the root node
cuts	average number of cuts generated
nodes	average number of nodes solved

		MM						LM						ML						LL						∅					
		opt^{tree}	$time$	opt^{root}	gap^{root}	$cuts$	$nodes$	opt^{tree}	$time$	opt^{root}	gap^{root}	$cuts$	$nodes$	opt^{tree}	$time$	opt^{root}	gap^{root}	$cuts$	$nodes$	opt^{tree}	$time$	opt^{root}	gap^{root}	$cuts$	$nodes$	opt^{tree}	$time$	opt^{root}	gap^{root}	$cuts$	$nodes$
A (orig.)	IMP^{max}_{min}	42	23	30	0.06	4	5	42	20	27	0.04	6	5	41	157	26	0.08	5	19	41	289	28	0.08	6	24	166	122	111	0.06	5	13
	$IMP\text{-}I^{max}$	42	25	30	0.07	21	10	42	15	25	0.06	17	7	40	192	23	0.11	31	38	41	205	24	0.08	22	39	165	109	102	0.08	23	23
	$IMP\text{-}I_{min}$	42	116	30	0.06	63	11	42	66	25	0.06	74	5	40	201	26	0.11	72	17	39	288	26	0.12	73	23	163	168	107	0.09	70	14
	$IMP\text{-}I$	42	103	30	0.07	76	10	41	138	24	0.07	79	8	40	242	23	0.13	91	20	39	286	22	0.12	82	24	162	192	99	0.10	82	15
B (4/3)	IMP^{max}_{min}	42	48	30	0.04	6	5	42	79	24	0.05	7	8	42	78	20	0.09	5	10	40	229	18	0.13	6	23	166	109	92	0.08	6	12
	$IMP\text{-}I^{max}$	42	29	23	0.06	35	9	42	39	23	0.07	24	12	41	155	18	0.13	41	36	40	267	17	0.15	28	46	165	122	81	0.10	32	26
	$IMP\text{-}I_{min}$	41	164	25	0.06	119	7	40	224	20	0.10	146	10	37	543	17	0.17	139	31	36	672	15	0.25	143	45	154	401	77	0.15	137	23
	$IMP\text{-}I$	40	218	20	0.08	144	10	40	257	20	0.12	161	13	37	599	15	0.21	166	31	35	728	15	0.27	156	56	152	451	70	0.17	157	28
C (5/3)	IMP^{max}_{min}	41	145	25	0.08	7	24	42	174	24	0.07	9	24	40	240	16	0.20	7	17	41	211	18	0.16	10	16	164	193	83	0.13	8	21
	$IMP\text{-}I^{max}$	41	134	18	0.14	55	33	42	75	21	0.10	31	25	39	308	12	0.26	65	62	41	133	14	0.19	35	30	163	163	65	0.18	47	38
	$IMP\text{-}I_{min}$	36	743	21	0.16	212	36	35	839	18	0.26	242	23	31	1048	11	0.40	221	50	33	981	13	0.41	222	52	135	903	63	0.31	224	41
	$IMP\text{-}I$	34	910	17	0.23	239	32	35	855	18	0.27	258	22	30	1237	8	0.48	253	64	32	1136	11	0.43	244	63	131	1034	54	0.35	248	46
D (6/3)	IMP^{max}_{min}	42	53	18	0.13	10	12	40	234	17	0.16	13	41	38	520	10	0.26	11	29	37	757	14	0.26	12	47	157	391	59	0.20	12	35
	$IMP\text{-}I^{max}$	42	92	14	0.17	73	32	42	185	16	0.20	53	63	36	701	8	0.50	91	145	39	544	12	0.35	55	108	159	381	50	0.30	68	93
	$IMP\text{-}I_{min}$	29	1313	15	0.31	345	24	26	1623	16	0.49	367	32	23	1862	7	0.82	336	80	22	1814	11	0.81	319	80	100	1653	49	0.61	342	58
	$IMP\text{-}I$	29	1346	13	0.36	372	22	23	1741	15	0.53	395	31	21	1971	7	0.97	372	80	22	1864	11	0.85	343	80	95	1730	46	0.68	371	57
∅	IMP^{max}_{min}	167	67	103	0.08	7	11	166	127	92	0.08	9	19	161	249	72	0.15	7	19	159	371	78	0.16	9	27	653	204	345	0.12	8	19
	$IMP\text{-}I^{max}$	167	70	85	0.11	46	21	168	79	85	0.11	31	27	156	339	61	0.25	57	70	161	287	67	0.19	35	55	652	194	298	0.17	42	43
	$IMP\text{-}I_{min}$	148	584	91	0.15	185	18	143	688	79	0.22	207	18	131	913	61	0.38	192	49	130	939	65	0.40	189	56	552	781	296	0.29	193	35
	$IMP\text{-}I$	145	644	80	0.19	208	18	139	748	77	0.25	223	18	128	1012	53	0.45	221	49	128	1003	59	0.42	206	56	540	852	269	0.33	214	35

Table 7: Results aggregated by subclass

Tables 8-11 present optimal solution/upper bound values for all instances together with the respective computation times of algorithms IMP_{min}^{max} and $IMP\text{-}I^{max}$. The meaning of the table entries are as follows:

inst	name of the instance
$opt/\overline{ub}$	value of optimal solution or best known upper bound (overlined)
$time_{IMP_{min}^{max}}$	computation time in seconds of algorithm IMP_{min}^{max}, *1h* if unable to solve instance within time limit
$time_{IMP\text{-}I^{max}}$	computation time in seconds of algorithm $IMP\text{-}I^{max}$, *1h* if unable to solve instance within time limit

MM				LM				ML				LL			
inst	$opt/\overline{ub}$	$time_{IMP^{max}_{min}}$	$time_{IMP\text{-}I^{max}}$	*inst*	$opt/\overline{ub}$	$time_{IMP^{max}_{min}}$	$time_{IMP\text{-}I^{max}}$	*inst*	$opt/\overline{ub}$	$time_{IMP^{max}_{min}}$	$time_{IMP\text{-}I^{max}}$	*inst*	$opt/\overline{ub}$	$time_{IMP^{max}_{min}}$	$time_{IMP\text{-}I^{max}}$
MM-A-a2-16	325.23	0.2	0.1	LM-A-a2-16	296.36	0.1	0.1	ML-A-a2-16	293.42	0.1	0.1	LL-A-a2-16	291.68	0.1	0.1
MM-A-a2-20	362.07	0.2	0.1	LM-A-a2-20	360.67	0.1	0.1	ML-A-a2-20	334.18	0.2	0.1	LL-A-a2-20	332.60	0.1	0.3
MM-A-a2-24	423.93	0.3	0.1	LM-A-a2-24	423.19	0.2	0.1	ML-A-a2-24	414.74	0.8	1.0	LL-A-a2-24	410.76	0.6	0.5
MM-A-a3-24	325.83	0.1	0.1	LM-A-a3-24	318.90	0.5	0.3	ML-A-a3-24	319.04	0.6	0.4	LL-A-a3-24	313.40	0.2	0.1
MM-A-a3-30	510.18	1.3	1.8	LM-A-a3-30	489.27	1.5	2.5	ML-A-a3-30	465.11	0.5	0.3	LL-A-a3-30	453.34	2.9	2.2
MM-A-a3-36	558.50	1.0	0.3	LM-A-a3-36	558.50	0.7	0.3	ML-A-a3-36	542.29	0.8	0.5	LL-A-a3-36	537.75	1.4	0.3
MM-A-a4-32	485.00	0.3	0.1	LM-A-a4-32	477.43	0.3	0.1	ML-A-a4-32	477.00	0.4	1.4	LL-A-a4-32	474.40	0.4	0.2
MM-A-a4-40	590.17	1.5	1.0	LM-A-a4-40	568.82	2.1	1.0	ML-A-a4-40	571.25	1.1	1.4	LL-A-a4-40	560.69	1.6	1.9
MM-A-a4-48	677.29	1.3	3.6	LM-A-a4-48	656.98	3.3	1.9	ML-A-a4-48	641.36	2.4	1.3	LL-A-a4-48	635.65	15.0	5.9
MM-A-a5-40	506.32	0.5	0.2	LM-A-a5-40	496.97	0.5	0.2	ML-A-a5-40	494.08	1.6	0.8	LL-A-a5-40	494.08	1.4	0.9
MM-A-a5-50	673.28	19.8	62.0	LM-A-a5-50	664.64	9.8	6.0	ML-A-a5-50	646.73	4.0	9.1	LL-A-a5-50	641.46	5.7	4.1
MM-A-a5-60	817.39	3.4	4.3	LM-A-a5-60	783.41	4.6	4.5	ML-A-a5-60	754.29	378.5	12.0	LL-A-a5-60	752.62	21.6	11.8
MM-A-a6-48	637.86	0.9	1.5	LM-A-a6-48	621.48	1.0	1.7	ML-A-a6-48	611.63	1.6	2.8	LL-A-a6-48	600.99	6.4	4.1
MM-A-a6-60	786.57	7.7	4.9	LM-A-a6-60	778.77	6.1	3.5	ML-A-a6-60	750.63	9.0	6.6	LL-A-a6-60	748.16	9.7	6.8
MM-A-a6-72	930.84	32.0	19.7	LM-A-a6-72	918.23	30.1	25.7	ML-A-a6-72	908.54	1836.7	*1h*	LL-A-a6-72	899.35	1876.7	2551.6
MM-A-a7-56	716.87	4.7	3.7	LM-A-a7-56	703.54	1.2	0.5	ML-A-a7-56	680.13	49.0	35.6	LL-A-a7-56	673.97	14.3	8.3
MM-A-a7-70	887.53	8.4	13.9	LM-A-a7-70	874.70	41.6	26.8	ML-A-a7-70	838.01	19.9	21.2	LL-A-a7-70	828.52	33.3	16.3
MM-A-a7-84	1031.31	332.2	588.4	LM-A-a7-84	1016.73	249.5	136.3	ML-A-a7-84	985.21	124.5	454.6	LL-A-a7-84	981.31	*1h*	*1h*
MM-A-a8-64	756.95	9.1	6.7	LM-A-a8-64	746.76	7.7	7.8	ML-A-a8-64	728.34	64.2	20.9	LL-A-a8-64	717.33	7.2	7.2
MM-A-a8-80	972.74	11.7	10.7	LM-A-a8-80	959.03	15.1	8.6	ML-A-a8-80	933.73	6.9	11.8	LL-A-a8-80	908.28	125.8	34.7
MM-A-a8-96	1186.95	307.0	125.6	LM-A-a8-96	1176.43	83.8	63.5	ML-A-a8-96	1149.95	*1h*	*1h*	LL-A-a8-96	1138.79	3050.3	2142.6
MM-A-b2-16	302.25	0.2	0.1	LM-A-b2-16	302.23	0.1	0.1	ML-A-b2-16	298.25	0.1	0.1	LL-A-b2-16	298.23	0.1	0.1
MM-A-b2-20	319.75	0.2	0.2	LM-A-b2-20	319.75	0.2	0.2	ML-A-b2-20	337.65	0.1	0.1	LL-A-b2-20	337.65	0.1	0.1
MM-A-b2-24	436.05	0.1	0.1	LM-A-b2-24	436.05	0.1	0.1	ML-A-b2-24	432.63	0.1	0.1	LL-A-b2-24	429.89	0.1	0.1
MM-A-b3-24	382.78	0.1	0.1	LM-A-b3-24	382.78	0.1	0.1	ML-A-b3-24	382.78	0.1	0.1	LL-A-b3-24	382.78	0.1	0.1
MM-A-b3-30	537.01	0.2	0.1	LM-A-b3-30	533.03	0.2	0.1	ML-A-b3-30	517.37	0.3	0.3	LL-A-b3-30	509.46	0.2	0.1
MM-A-b3-36	584.43	0.3	0.1	LM-A-b3-36	593.13	0.5	0.2	ML-A-b3-36	583.97	1.2	1.0	LL-A-b3-36	588.75	0.5	0.2
MM-A-b4-32	543.40	0.3	0.4	LM-A-b4-32	539.80	0.2	0.1	ML-A-b4-32	537.68	0.2	0.1	LL-A-b4-32	534.18	0.2	0.1
MM-A-b4-40	666.72	1.2	1.0	LM-A-b4-40	676.38	3.3	4.7	ML-A-b4-40	657.24	0.4	0.2	LL-A-b4-40	667.04	0.6	0.2
MM-A-b4-48	693.59	2.6	1.4	LM-A-b4-48	700.44	3.6	2.2	ML-A-b4-48	691.15	2.1	1.4	LL-A-b4-48	697.87	6.1	2.4
MM-A-b5-40	633.24	3.8	2.5	LM-A-b5-40	627.71	2.8	2.0	ML-A-b5-40	623.83	1.6	1.3	LL-A-b5-40	618.32	0.5	0.2
MM-A-b5-50	805.14	1.1	1.9	LM-A-b5-50	798.52	1.7	4.4	ML-A-b5-50	788.38	1.7	2.5	LL-A-b5-50	781.30	1.9	3.8
MM-A-b5-60	972.70	12.1	9.6	LM-A-b5-60	941.00	11.9	38.6	ML-A-b5-60	937.18	13.8	15.1	LL-A-b5-60	930.51	15.8	12.0
MM-A-b6-48	747.34	1.3	1.6	LM-A-b6-48	729.66	0.7	0.5	ML-A-b6-48	733.57	1.3	1.6	LL-A-b6-48	718.86	0.6	0.2
MM-A-b6-60	898.04	2.0	3.7	LM-A-b6-60	889.33	1.7	0.7	ML-A-b6-60	895.06	14.3	15.4	LL-A-b6-60	884.08	2.0	0.8
MM-A-b6-72	1015.55	48.2	41.6	LM-A-b6-72	1000.44	203.7	113.6	ML-A-b6-72	999.64	53.2	49.9	LL-A-b6-72	990.79	110.1	47.7
MM-A-b7-56	865.07	1.1	0.4	LM-A-b7-56	853.32	3.8	2.6	ML-A-b7-56	843.58	3.1	4.3	LL-A-b7-56	834.60	16.1	13.5
MM-A-b7-70	982.59	10.7	16.6	LM-A-b7-70	950.48	5.7	9.4	ML-A-b7-70	973.08	6.7	15.9	LL-A-b7-70	946.36	8.1	17.1
MM-A-b7-84	1283.07	106.1	77.2	LM-A-b7-84	1272.57	43.8	34.9	ML-A-b7-84	1261.89	326.5	119.0	LL-A-b7-84	1240.59	3162.5	59.6
MM-A-b8-64	888.80	7.8	5.2	LM-A-b8-64	869.05	6.1	4.2	ML-A-b8-64	880.97	9.1	8.1	LL-A-b8-64	861.49	5.5	4.1
MM-A-b8-80	1105.35	3.7	13.1	LM-A-b8-80	1088.24	10.5	18.2	ML-A-b8-80	1090.82	3.9	14.3	LL-A-b8-80	1074.06	9.5	14.7
MM-A-b8-96	1240.16	22.7	11.7	LM-A-b8-96	1236.96	90.6	108.2	ML-A-b8-96	1227.98	39.6	30.6	LL-A-b8-96	1215.04	15.3	31.8

Table 8: Results for class A (orig.) instances

MM				LM				ML				LL			
inst	$opt/\overline{ub}$	$time_{IMP^{max}_{min}}$	$time_{IMP\text{-}I^{max}}$	*inst*	$opt/\overline{ub}$	$time_{IMP^{max}_{min}}$	$time_{IMP\text{-}I^{max}}$	*inst*	$opt/\overline{ub}$	$time_{IMP^{max}_{min}}$	$time_{IMP\text{-}I^{max}}$	*inst*	$opt/\overline{ub}$	$time_{IMP^{max}_{min}}$	$time_{IMP\text{-}I^{max}}$
MM-B-a2-16	302.60	0.2	0.1	LM-B-a2-16	296.36	0.1	0.1	ML-B-a2-16	293.42	0.1	0.1	LL-B-a2-16	291.68	0.1	0.1
MM-B-a2-20	358.82	0.1	0.1	LM-B-a2-20	358.82	0.1	0.1	ML-B-a2-20	333.61	0.1	0.1	LL-B-a2-20	332.02	0.5	0.5
MM-B-a2-24	414.96	0.3	0.5	LM-B-a2-24	413.00	0.9	0.7	ML-B-a2-24	410.60	1.6	1.5	LL-B-a2-24	406.61	1.3	0.8
MM-B-a3-24	303.23	0.2	0.1	LM-B-a3-24	303.23	0.2	0.1	ML-B-a3-24	296.29	0.3	0.1	LL-B-a3-24	296.29	0.2	0.1
MM-B-a3-30	490.31	2.1	1.6	LM-B-a3-30	481.28	0.4	1.5	ML-B-a3-30	455.16	4.3	3.2	LL-B-a3-30	449.74	6.0	2.6
MM-B-a3-36	544.06	1.6	0.9	LM-B-a3-36	534.09	2.4	1.0	ML-B-a3-36	529.88	2.8	2.4	LL-B-a3-36	520.14	2.2	2.1
MM-B-a4-32	484.55	0.4	1.4	LM-B-a4-32	470.24	0.3	0.1	ML-B-a4-32	461.66	0.6	0.2	LL-B-a4-32	454.77	1.7	1.3
MM-B-a4-40	571.27	0.9	0.4	LM-B-a4-40	568.82	4.6	2.1	ML-B-a4-40	558.78	1.5	2.5	LL-B-a4-40	556.46	25.3	31.9
MM-B-a4-48	667.10	22.9	5.4	LM-B-a4-48	653.31	22.0	4.4	ML-B-a4-48	627.75	3.8	1.9	LL-B-a4-48	624.02	25.4	5.7
MM-B-a5-40	498.33	1.6	1.2	LM-B-a5-40	496.79	1.5	0.8	ML-B-a5-40	483.97	1.7	1.1	LL-B-a5-40	483.24	4.1	2.6
MM-B-a5-50	655.30	3.9	2.2	LM-B-a5-50	646.39	7.7	5.4	ML-B-a5-50	633.65	12.1	25.6	LL-B-a5-50	627.99	78.1	24.6
MM-B-a5-60	778.20	11.3	10.1	LM-B-a5-60	751.73	7.3	4.8	ML-B-a5-60	731.86	7.0	6.7	LL-B-a5-60	730.95	59.1	32.6
MM-B-a6-48	612.30	1.5	2.0	LM-B-a6-48	610.23	3.2	1.5	ML-B-a6-48	585.35	6.0	4.9	LL-B-a6-48	576.07	3.0	2.8
MM-B-a6-60	768.84	11.7	10.1	LM-B-a6-60	755.64	11.1	7.7	ML-B-a6-60	728.62	38.0	28.0	LL-B-a6-60	714.67	14.6	6.8
MM-B-a6-72	907.44	26.3	19.0	LM-B-a6-72	897.02	18.6	14.6	ML-B-a6-72	876.11	407.5	298.1	LL-B-a6-72	865.49	42.3	40.1
MM-B-a7-56	692.13	4.1	5.1	LM-B-a7-56	691.11	7.7	17.8	ML-B-a7-56	666.67	18.1	38.2	LL-B-a7-56	639.71	15.2	7.8
MM-B-a7-70	866.44	12.2	15.4	LM-B-a7-70	853.72	16.5	6.2	ML-B-a7-70	824.92	29.5	36.0	LL-B-a7-70	813.35	70.8	49.4
MM-B-a7-84	1007.64	79.9	94.2	LM-B-a7-84	989.92	73.5	41.4	ML-B-a7-84	958.41	664.6	764.4	LL-B-a7-84	952.52	*1h*	*1h*
MM-B-a8-64	737.25	7.7	8.4	LM-B-a8-64	726.35	27.6	14.8	ML-B-a8-64	703.74	22.1	15.7	LL-B-a8-64	700.10	32.8	28.8
MM-B-a8-80	958.43	38.4	18.8	LM-B-a8-80	933.74	27.4	20.7	ML-B-a8-80	900.45	111.1	708.4	LL-B-a8-80	881.77	85.9	130.2
MM-B-a8-96	1162.61	1576.2	780.8	LM-B-a8-96	1154.52	2851.8	1212.3	ML-B-a8-96	1114.48	938.3	*1h*	LL-B-a8-96	1103.35	1597.5	3429.5
MM-B-b2-16	289.24	0.1	0.1	LM-B-b2-16	289.24	0.2	0.2	ML-B-b2-16	289.22	0.4	0.3	LL-B-b2-16	289.22	0.3	0.3
MM-B-b2-20	323.71	0.2	0.1	LM-B-b2-20	323.71	0.1	0.1	ML-B-b2-20	323.71	0.2	0.1	LL-B-b2-20	323.71	0.2	0.1
MM-B-b2-24	458.96	0.3	0.1	LM-B-b2-24	458.69	0.2	0.1	ML-B-b2-24	454.60	0.2	0.1	LL-B-b2-24	451.81	0.3	0.1
MM-B-b3-24	361.08	0.1	0.1	LM-B-b3-24	360.24	0.1	0.1	ML-B-b3-24	361.08	0.1	0.1	LL-B-b3-24	360.24	0.1	0.1
MM-B-b3-30	527.64	0.4	0.2	LM-B-b3-30	521.42	0.4	0.1	ML-B-b3-30	504.37	0.3	0.5	LL-B-b3-30	501.27	0.3	0.1
MM-B-b3-36	570.20	0.6	0.3	LM-B-b3-36	553.04	0.6	0.2	ML-B-b3-36	570.20	0.5	0.3	LL-B-b3-36	553.04	0.6	0.3
MM-B-b4-32	487.52	0.3	0.8	LM-B-b4-32	481.45	0.3	0.2	ML-B-b4-32	456.45	0.8	0.7	LL-B-b4-32	456.26	0.9	0.6
MM-B-b4-40	618.08	1.5	3.0	LM-B-b4-40	611.16	1.7	1.7	ML-B-b4-40	609.67	1.9	3.7	LL-B-b4-40	600.16	1.4	0.9
MM-B-b4-48	656.77	6.8	8.0	LM-B-b4-48	637.10	3.7	2.1	ML-B-b4-48	643.64	8.5	7.1	LL-B-b4-48	629.47	2.9	0.9
MM-B-b5-40	603.96	1.7	0.9	LM-B-b5-40	599.88	1.3	0.8	ML-B-b5-40	579.84	2.1	2.9	LL-B-b5-40	579.40	2.3	2.0
MM-B-b5-50	755.61	5.3	4.3	LM-B-b5-50	736.76	14.3	25.0	ML-B-b5-50	719.21	48.0	121.2	LL-B-b5-50	715.82	139.0	47.3
MM-B-b5-60	871.92	6.8	11.6	LM-B-b5-60	866.01	6.9	30.6	ML-B-b5-60	857.14	139.7	306.1	LL-B-b5-60	846.67	9.4	26.7
MM-B-b6-48	706.54	1.7	3.5	LM-B-b6-48	696.19	1.8	1.6	ML-B-b6-48	680.92	0.9	1.8	LL-B-b6-48	677.45	3.0	3.4
MM-B-b6-60	844.48	2.1	4.6	LM-B-b6-60	839.92	36.1	17.6	ML-B-b6-60	829.15	13.4	7.1	LL-B-b6-60	824.66	29.8	9.1
MM-B-b6-72	964.04	68.3	94.1	LM-B-b6-72	954.51	18.6	17.5	ML-B-b6-72	930.21	32.3	50.3	LL-B-b6-72	925.96	51.9	51.1
MM-B-b7-56	764.61	3.6	3.9	LM-B-b7-56	758.26	5.7	3.9	ML-B-b7-56	736.85	4.8	2.9	LL-B-b7-56	736.77	6.3	3.6
MM-B-b7-70	880.91	4.7	4.6	LM-B-b7-70	861.72	10.7	9.0	ML-B-b7-70	870.40	22.1	14.6	LL-B-b7-70	850.36	56.9	27.1
MM-B-b7-84	1173.74	20.8	11.2	LM-B-b7-84	1157.16	54.2	40.1	ML-B-b7-84	1137.84	185.2	183.6	LL-B-b7-84	1125.09	31.1	21.5
MM-B-b8-64	791.96	22.8	23.4	LM-B-b8-64	781.98	11.6	15.1	ML-B-b8-64	773.99	3.7	3.9	LL-B-b8-64	761.51	16.0	12.4
MM-B-b8-80	1011.77	30.1	18.1	LM-B-b8-80	997.54	30.1	22.9	ML-B-b8-80	995.44	17.7	21.5	LL-B-b8-80	986.23	14.4	13.1
MM-B-b8-96	1151.96	18.4	57.8	LM-B-b8-96	1147.03	52.3	80.2	ML-B-b8-96	1111.63	540.2	233.4	LL-B-b8-96	$\overline{1107.70}$	*1h*	*1h*

Table 9: Results for class B (4/3) instances

MM				LM				ML				LL			
$inst$	$opt/\overline{ub}$	$time_{IMP^{max}_{min}}$	$time_{IMP\text{-}I^{max}}$	$inst$	$opt/\overline{ub}$	$time_{IMP^{max}_{min}}$	$time_{IMP\text{-}I^{max}}$	$inst$	$opt/\overline{ub}$	$time_{IMP^{max}_{min}}$	$time_{IMP\text{-}I^{max}}$	$inst$	$opt/\overline{ub}$	$time_{IMP^{max}_{min}}$	$time_{IMP\text{-}I^{max}}$
MM-C-a2-16	300.86	0.1	0.1	LM-C-a2-16	296.36	0.1	0.1	ML-C-a2-16	289.40	0.2	0.1	LL-C-a2-16	289.40	0.2	0.1
MM-C-a2-20	354.84	0.3	1.1	LM-C-a2-20	351.49	0.2	0.1	ML-C-a2-20	319.13	0.5	0.5	LL-C-a2-20	315.59	0.6	0.5
MM-C-a2-24	409.20	1.4	1.1	LM-C-a2-24	407.46	2.4	1.7	ML-C-a2-24	405.76	4.1	5.9	LL-C-a2-24	403.94	7.0	4.3
MM-C-a3-24	303.10	0.2	1.7	LM-C-a3-24	299.26	0.2	0.4	ML-C-a3-24	296.02	0.3	1.3	LL-C-a3-24	295.43	0.3	0.7
MM-C-a3-30	471.48	0.4	0.7	LM-C-a3-30	462.89	1.8	2.2	ML-C-a3-30	443.25	3.7	1.9	LL-C-a3-30	436.71	0.9	1.2
MM-C-a3-36	535.36	2.0	1.6	LM-C-a3-36	501.05	2.2	1.1	ML-C-a3-36	523.09	3.6	2.0	LL-C-a3-36	476.85	1.4	0.9
MM-C-a4-32	472.33	0.8	2.2	LM-C-a4-32	462.41	1.0	4.3	ML-C-a4-32	456.97	3.8	6.0	LL-C-a4-32	449.06	3.1	2.7
MM-C-a4-40	557.31	2.6	2.0	LM-C-a4-40	540.58	2.6	2.4	ML-C-a4-40	527.18	35.5	53.9	LL-C-a4-40	520.10	7.9	3.9
MM-C-a4-48	651.53	3.8	3.1	LM-C-a4-48	641.98	7.8	4.5	ML-C-a4-48	621.44	95.8	26.4	LL-C-a4-48	612.24	68.3	23.4
MM-C-a5-40	493.34	0.6	0.2	LM-C-a5-40	491.02	2.0	1.3	ML-C-a5-40	468.99	2.0	1.4	LL-C-a5-40	468.99	2.4	1.4
MM-C-a5-50	647.08	11.5	17.1	LM-C-a5-50	635.09	13.8	11.9	ML-C-a5-50	623.51	100.1	130.3	LL-C-a5-50	614.56	128.2	281.4
MM-C-a5-60	770.48	14.5	7.1	LM-C-a5-60	742.39	6.0	5.9	ML-C-a5-60	716.03	347.4	78.6	LL-C-a5-60	713.79	47.1	20.3
MM-C-a6-48	609.34	1.6	2.8	LM-C-a6-48	606.39	4.4	2.5	ML-C-a6-48	571.69	101.0	30.1	LL-C-a6-48	570.19	236.0	31.9
MM-C-a6-60	754.46	65.9	25.1	LM-C-a6-60	744.35	31.6	43.4	ML-C-a6-60	713.59	19.8	17.8	LL-C-a6-60	707.32	27.1	24.5
MM-C-a6-72	895.60	25.1	28.8	LM-C-a6-72	882.09	20.2	87.0	ML-C-a6-72	861.28	132.8	712.0	LL-C-a6-72	852.35	815.3	136.4
MM-C-a7-56	684.15	42.5	54.3	LM-C-a7-56	668.72	8.3	7.0	ML-C-a7-56	646.31	17.9	13.4	LL-C-a7-56	630.31	28.2	23.9
MM-C-a7-70	857.34	111.2	449.0	LM-C-a7-70	842.63	26.5	31.0	ML-C-a7-70	807.59	24.9	50.9	LL-C-a7-70	794.65	33.1	20.2
MM-C-a7-84	995.37	1102.6	691.5	LM-C-a7-84	975.07	2356.4	569.3	ML-C-a7-84	932.83	65.8	128.1	LL-C-a7-84	926.54	463.2	141.5
MM-C-a8-64	716.43	3.6	17.3	LM-C-a8-64	699.27	17.2	10.2	ML-C-a8-64	695.34	488.9	259.5	LL-C-a8-64	687.57	389.3	82.4
MM-C-a8-80	924.19	119.5	63.7	LM-C-a8-80	912.41	21.6	34.4	ML-C-a8-80	880.66	39.5	83.4	LL-C-a8-80	867.23	394.9	448.7
MM-C-a8-96	1144.32	60.8	77.8	LM-C-a8-96	1133.72	968.7	180.9	ML-C-a8-96	$\overline{1101.72}$	*1h*	*1h*	LL-C-a8-96	1087.91	*1h*	*1h*
MM-C-b2-16	279.95	0.3	0.2	LM-C-b2-16	279.64	0.5	0.6	ML-C-b2-16	278.01	0.4	0.6	LL-C-b2-16	274.19	0.1	0.1
MM-C-b2-20	315.00	0.1	0.1	LM-C-b2-20	315.00	0.1	0.1	ML-C-b2-20	315.00	0.4	0.5	LL-C-b2-20	314.13	0.4	0.3
MM-C-b2-24	417.66	0.2	0.1	LM-C-b2-24	417.52	0.2	0.1	ML-C-b2-24	412.49	0.6	0.5	LL-C-b2-24	410.13	0.2	0.2
MM-C-b3-24	353.81	0.3	0.5	LM-C-b3-24	351.64	0.1	0.3	ML-C-b3-24	353.81	0.3	0.8	LL-C-b3-24	351.52	0.1	0.3
MM-C-b3-30	467.01	0.7	0.4	LM-C-b3-30	467.01	0.8	0.5	ML-C-b3-30	456.89	0.4	0.5	LL-C-b3-30	455.59	0.5	0.2
MM-C-b3-36	549.85	0.5	0.2	LM-C-b3-36	542.24	0.6	0.3	ML-C-b3-36	533.92	0.6	0.3	LL-C-b3-36	525.43	0.6	0.2
MM-C-b4-32	465.99	0.8	1.0	LM-C-b4-32	465.99	1.1	0.9	ML-C-b4-32	450.17	7.8	6.0	LL-C-b4-32	449.79	11.2	6.6
MM-C-b4-40	595.06	1.9	6.1	LM-C-b4-40	580.10	1.6	1.2	ML-C-b4-40	568.19	2.0	1.5	LL-C-b4-40	555.57	1.9	1.3
MM-C-b4-48	633.02	2.7	3.7	LM-C-b4-48	619.27	2.2	4.5	ML-C-b4-48	624.58	15.7	12.0	LL-C-b4-48	600.50	6.3	4.9
MM-C-b5-40	563.53	3.1	4.4	LM-C-b5-40	560.44	0.9	2.4	ML-C-b5-40	552.38	3.4	8.4	LL-C-b5-40	550.34	7.0	7.5
MM-C-b5-50	688.43	11.2	8.4	LM-C-b5-50	686.27	12.6	6.6	ML-C-b5-50	665.24	46.7	21.2	LL-C-b5-50	662.70	47.9	14.6
MM-C-b5-60	815.30	9.4	9.8	LM-C-b5-60	811.53	8.2	17.4	ML-C-b5-60	788.81	53.0	22.6	LL-C-b5-60	780.49	7.3	37.4
MM-C-b6-48	650.88	0.9	2.5	LM-C-b6-48	641.55	4.4	3.6	ML-C-b6-48	635.37	4.5	17.2	LL-C-b6-48	626.08	16.1	10.6
MM-C-b6-60	801.09	2.2	6.0	LM-C-b6-60	780.45	2.9	1.6	ML-C-b6-60	773.95	6.9	12.7	LL-C-b6-60	759.93	14.4	11.3
MM-C-b6-72	919.55	296.1	253.0	LM-C-b6-72	914.27	25.6	17.2	ML-C-b6-72	881.75	62.8	59.8	LL-C-b6-72	879.38	89.9	23.9
MM-C-b7-56	705.29	12.0	6.1	LM-C-b7-56	697.02	26.3	14.2	ML-C-b7-56	676.10	7.5	8.0	LL-C-b7-56	674.76	13.4	12.2
MM-C-b7-70	852.49	8.3	10.1	LM-C-b7-70	832.61	42.6	58.6	ML-C-b7-70	829.00	8.5	9.6	LL-C-b7-70	810.92	27.6	21.7
MM-C-b7-84	1103.57	445.8	192.4	LM-C-b7-84	1086.43	37.7	48.6	ML-C-b7-84	1061.32	146.6	329.5	LL-C-b7-84	1054.14	221.5	107.4
MM-C-b8-64	717.73	3.8	9.8	LM-C-b8-64	709.66	7.2	4.6	ML-C-b8-64	705.86	22.2	26.9	LL-C-b8-64	690.12	36.5	19.2
MM-C-b8-80	943.70	*1h*	*1h*	LM-C-b8-80	937.16	3595.3	1960.3	ML-C-b8-80	925.53	999.9	*1h*	LL-C-b8-80	915.44	363.9	170.9
MM-C-b8-96	1096.43	113.8	56.1	LM-C-b8-96	1087.88	62.7	24.6	ML-C-b8-96	1047.67	*1h*	*1h*	LL-C-b8-96	1034.18	1731.5	300.6

Table 10: Results for class C (5/3) instances

MM				LM				ML				LL			
inst	$opt/\overline{ub}$	$time_{IMP^{max}_{min}}$	$time_{IMP\text{-}I^{max}}$	*inst*	$opt/\overline{ub}$	$time_{IMP^{max}_{min}}$	$time_{IMP\text{-}I^{max}}$	*inst*	$opt/\overline{ub}$	$time_{IMP^{max}_{min}}$	$time_{IMP\text{-}I^{max}}$	*inst*	$opt/\overline{ub}$	$time_{IMP^{max}_{min}}$	$time_{IMP\text{-}I^{max}}$
MM-D-a2-16	296.46	0.1	0.1	LM-D-a2-16	285.89	0.1	0.1	ML-D-a2-16	280.61	0.1	0.2	LL-D-a2-16	271.94	0.1	0.1
MM-D-a2-20	351.49	0.2	0.2	LM-D-a2-20	351.49	0.6	0.5	ML-D-a2-20	308.81	0.2	0.4	LL-D-a2-20	304.57	0.5	0.4
MM-D-a2-24	407.54	2.0	2.9	LM-D-a2-24	407.46	3.5	2.4	ML-D-a2-24	401.32	4.7	4.3	LL-D-a2-24	382.74	3.9	2.6
MM-D-a3-24	299.26	0.2	0.9	LM-D-a3-24	297.77	0.3	0.4	ML-D-a3-24	293.19	1.9	1.4	LL-D-a3-24	289.57	1.6	0.7
MM-D-a3-30	461.19	3.6	8.1	LM-D-a3-30	453.45	4.0	4.3	ML-D-a3-30	432.15	1.2	5.8	LL-D-a3-30	417.82	1.7	0.4
MM-D-a3-36	502.16	2.5	1.6	LM-D-a3-36	500.19	2.2	1.2	ML-D-a3-36	475.71	19.8	0.7	LL-D-a3-36	471.16	1.5	0.5
MM-D-a4-32	458.70	3.7	5.0	LM-D-a4-32	457.19	9.8	6.1	ML-D-a4-32	439.19	5.3	8.5	LL-D-a4-32	435.87	3.3	3.8
MM-D-a4-40	549.36	3.4	3.7	LM-D-a4-40	518.20	8.8	4.2	ML-D-a4-40	521.95	46.1	60.3	LL-D-a4-40	503.98	31.9	7.5
MM-D-a4-48	638.93	86.7	45.9	LM-D-a4-48	615.97	5.0	2.8	ML-D-a4-48	603.55	414.8	137.1	LL-D-a4-48	599.43	1606.6	471.1
MM-D-a5-40	471.86	1.9	1.1	LM-D-a5-40	467.48	1.0	0.3	ML-D-a5-40	450.48	11.5	3.8	LL-D-a5-40	443.19	1.8	0.5
MM-D-a5-50	629.65	17.9	10.8	LM-D-a5-50	619.63	42.1	10.4	ML-D-a5-50	595.16	22.8	10.1	LL-D-a5-50	585.90	1655.6	329.9
MM-D-a5-60	759.70	36.2	14.5	LM-D-a5-60	735.26	45.3	15.3	ML-D-a5-60	699.20	109.4	37.8	LL-D-a5-60	689.85	135.7	44.9
MM-D-a6-48	589.17	9.6	9.7	LM-D-a6-48	575.93	2.8	3.9	ML-D-a6-48	559.60	404.2	19.0	LL-D-a6-48	558.59	203.8	38.0
MM-D-a6-60	744.79	58.9	61.8	LM-D-a6-60	734.92	44.7	31.4	ML-D-a6-60	704.73	53.3	122.8	LL-D-a6-60	697.59	75.1	47.8
MM-D-a6-72	871.71	32.8	32.6	LM-D-a6-72	864.78	38.6	19.0	ML-D-a6-72	853.62	*1h*	*1h*	LL-D-a6-72	839.37	3333.2	1974.1
MM-D-a7-56	664.87	5.4	3.6	LM-D-a7-56	659.15	31.5	18.7	ML-D-a7-56	632.79	32.9	25.1	LL-D-a7-56	619.50	94.9	50.5
MM-D-a7-70	836.38	29.6	125.6	LM-D-a7-70	827.30	77.0	39.4	ML-D-a7-70	792.07	52.3	85.6	LL-D-a7-70	775.68	47.6	56.6
MM-D-a7-84	968.87	232.9	1150.2	LM-D-a7-84	953.92	*1h*	2680.8	ML-D-a7-84	$\overline{927.15}$	*1h*	*1h*	LL-D-a7-84	909.82	*1h*	*1h*
MM-D-a8-64	707.89	126.3	153.6	LM-D-a8-64	693.22	32.6	29.3	ML-D-a8-64	685.05	304.5	153.2	LL-D-a8-64	677.88	669.2	1197.0
MM-D-a8-80	910.29	46.1	73.7	LM-D-a8-80	896.52	38.7	39.2	ML-D-a8-80	862.07	481.5	477.3	LL-D-a8-80	847.78	1142.3	666.1
MM-D-a8-96	1117.93	67.2	34.5	LM-D-a8-96	1108.70	430.6	87.5	ML-D-a8-96	$\overline{1070.58}$	*1h*	*1h*	LL-D-a8-96	1057.65	*1h*	*1h*
MM-D-b2-16	279.95	0.2	0.3	LM-D-b2-16	274.50	0.1	0.2	ML-D-b2-16	274.50	0.1	0.1	LL-D-b2-16	270.99	0.2	0.1
MM-D-b2-20	315.00	0.2	0.1	LM-D-b2-20	313.11	0.5	0.8	ML-D-b2-20	313.31	0.6	0.6	LL-D-b2-20	306.23	0.2	0.5
MM-D-b2-24	406.83	0.3	0.6	LM-D-b2-24	404.87	0.2	0.1	ML-D-b2-24	394.86	2.4	8.3	LL-D-b2-24	384.27	0.6	0.2
MM-D-b3-24	334.94	0.1	0.1	LM-D-b3-24	329.80	0.1	0.3	ML-D-b3-24	329.68	0.5	0.3	LL-D-b3-24	328.50	0.5	0.3
MM-D-b3-30	465.34	0.4	1.5	LM-D-b3-30	464.84	0.5	0.2	ML-D-b3-30	447.15	1.7	1.1	LL-D-b3-30	446.76	2.8	1.7
MM-D-b3-36	542.92	1.1	0.8	LM-D-b3-36	542.92	1.2	1.0	ML-D-b3-36	520.12	1.3	0.7	LL-D-b3-36	519.72	1.5	0.6
MM-D-b4-32	439.65	0.3	0.6	LM-D-b4-32	435.90	1.4	0.9	ML-D-b4-32	419.33	1.5	2.5	LL-D-b4-32	419.21	2.4	2.5
MM-D-b4-40	556.70	1.7	1.3	LM-D-b4-40	522.69	2.1	1.5	ML-D-b4-40	528.30	2.4	2.3	LL-D-b4-40	502.92	5.1	2.7
MM-D-b4-48	628.44	5.1	7.4	LM-D-b4-48	615.54	4.7	5.9	ML-D-b4-48	606.67	76.7	288.6	LL-D-b4-48	584.89	7.7	13.6
MM-D-b5-40	554.12	30.1	13.8	LM-D-b5-40	549.60	5.4	3.5	ML-D-b5-40	534.30	22.3	26.3	LL-D-b5-40	530.71	12.0	11.1
MM-D-b5-50	668.62	13.3	22.0	LM-D-b5-50	664.22	58.7	60.8	ML-D-b5-50	642.21	246.3	2364.5	LL-D-b5-50	629.16	46.6	23.3
MM-D-b5-60	787.38	7.9	9.1	LM-D-b5-60	785.21	10.4	38.3	ML-D-b5-60	745.49	23.3	49.8	LL-D-b5-60	742.25	304.6	414.9
MM-D-b6-48	627.30	14.6	21.6	LM-D-b6-48	616.86	24.1	10.7	ML-D-b6-48	606.24	194.4	1631.5	LL-D-b6-48	590.15	72.2	46.8
MM-D-b6-60	748.29	3.5	12.9	LM-D-b6-60	742.45	69.4	43.8	ML-D-b6-60	733.29	59.9	163.9	LL-D-b6-60	726.76	*1h*	2496.8
MM-D-b6-72	895.50	114.5	184.3	LM-D-b6-72	889.12	470.1	134.6	ML-D-b6-72	851.35	2051.7	1623.9	LL-D-b6-72	840.78	2653.0	688.6
MM-D-b7-56	682.37	12.6	41.2	LM-D-b7-56	674.81	154.2	41.0	ML-D-b7-56	635.15	125.8	64.3	LL-D-b7-56	630.27	69.9	32.0
MM-D-b7-70	823.95	59.5	158.7	LM-D-b7-70	805.16	172.8	104.9	ML-D-b7-70	777.62	44.1	35.4	LL-D-b7-70	760.55	42.4	18.7
MM-D-b7-84	1057.05	773.2	773.4	LM-D-b7-84	1045.84	396.1	70.1	ML-D-b7-84	1016.84	*1h*	*1h*	LL-D-b7-84	1012.45	*1h*	2775.8
MM-D-b8-64	704.70	29.9	103.9	LM-D-b8-64	695.72	*1h*	3405.6	ML-D-b8-64	683.44	439.5	406.0	LL-D-b8-64	669.07	764.4	268.1
MM-D-b8-80	906.13	42.2	447.5	LM-D-b8-80	897.14	106.5	568.6	ML-D-b8-80	869.68	1690.1	*1h*	LL-D-b8-80	$\overline{861.25}$	*1h*	*1h*
MM-D-b8-96	1079.80	355.9	323.1	LM-D-b8-96	1068.61	341.8	294.0	ML-D-b8-96	1021.89	486.0	*1h*	LL-D-b8-96	1001.47	779.8	365.1

Table 11: Results for class D (6/3) instances