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Solving Static and Dynamic Seat Allocation Models with Two Substitutable Resources

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Abstract

Airlines traditionally serve customer segments with demand for specific products by selling a seat on a particular flight leg in combination with a fare class. Products that are determined to target flexible customer segments are known as flexible products. They combine a fare class with one unit of a predefined set of substitutable resources. In such a revenue management setting, controlling the process of selling shared seat inventory includes assigning the accepted flexible customer requests to resources. The purpose of this paper is to analyze the common structure of optimal accept-and-assign decisions in static and dynamic seat-allocation models, which are wide-spread in literature and practice. For flexible products with two substitutable resources, multiple fares and online assignment, our results provide structural and computational insights in two respects: First, we show that componentwise concavity and submodularity are sufficient to derive optimal booking policies. It was known before that concavity, submodularity, and subconcavity are sufficient in the case of dynamic value functions. Thus, our results are based on less restrictive assumptions and apply also to static value functions. We derive a general optimal booking policy based on four monotonic switching curves. Second, we analyze the computational complexity of computing value functions and optimal booking decisions, and we herewith show that the respective dynamic programs can be solved efficiently.

1. Introduction and literature review

Airlines commonly enhance revenues by offering heterogeneous customer segments various fare classes, i.e., different combinations of a sales price and some sales restrictions for a seat in the same cabin. In the short run, when the set of products and the flight schedule are fixed, revenue management (RM) is synonymous with controlling the process of selling shared, perishable inventory subject to limited seat capacity and uncertain demand patterns. A resource is said to perish at the end of the sales horizon, meaning that any vacant seat is worthless once the plane has departed. A reasonable approach to the basic seat allocation problem is to balance two types of opportunity costs in order to maximize expected revenue: On the one hand, accepting early low-fare customers might increase the risk of spilling future high-fare customers owing to lack of capacity. On the other hand, rejecting too many customers and holding perishable inventory bears the risk of being stuck with unsold seats.

Airlines traditionally address customer segments with demand for specific products or services by attaching fare classes to one unit of a specific resource, i.e., one seat on a particular flight leg. If the airline accepts a request for such a *specific product*, seat availability will immediately reduce at the product and resource level. In recent years, service providers have been observed launching *flexible products* that combine fare classes with one unit of a predefined set of substitutable resources (Gallego and Phillips, 2004). For instance, an airline offers flights from Frankfurt International Airport (FRA) to several of London's five major airports. If the airline sells a FRA-London ticket to a flexible passenger, the airline still has to choose

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one particular FRA-London leg. Providers who want to control the sale of flexible products end up solving both an acceptance and an assignment problem.

In a network RM setting, a flexible product represents two or more substitutable bundles of complementary resources, e.g., connecting flight legs or consecutive overnight stays (Gallego *et al.*, 2004). Talluri (2001) developed a solution approach to a similar network RM problem assuming some known fraction of "routable" demand for each origin-destination fare class combination. Apart from routing-related product attributes, airlines have tried to hide outbound and return flight dates from the buyer of a flexible product (Post, 2010). RM under substitution is also a matter in non-airline industries, e.g., broadcasting (Kimms and Müller-Bungart, 2007) and car rental (Steinhardt and Gönsch, 2012). Gallego and Phillips (2004) reported other applications in the air cargo and hospitality industries.

We confine the following literature review to papers that deal with the RM of substitutable resources. Initial studies attempted to justify the use of flexible products by identifying different sources of revenue enhancements. Flexible products potentially attract new customers as a result of a more differentiated segmentation and pricing, but might increase buy-down behaviour to some extent (Gallego and Phillips, 2004). From the standpoint of capacity allocation, exploiting the possibility to assign flexible customers to lower utilized flights in order to protect highly utilized flights is likely to serve full-fare passengers who would have been denied otherwise (Talluri, 2001; Gallego and Phillips, 2004; Petrick *et al.*, 2012). Also, assigning accepted customers later than at the actual booking date can gain additional revenue (Gallego and Phillips, 2004; Petrick *et al.*, 2010). Roughly speaking, the additional degree of freedom in controlling how and when customers are assigned to resources provides a hedge against uncertainty about high-value demand. Chen *et al.* (2010) discussed a more subtle revenue impact that stems from an arbitrary substitution order as opposed to a restricted substitution order. Some authors have pointed out that substitution between different types of resources can release further revenue opportunities. Particularly, the trade-off between downgrading some customers at a cost while upgrading others at a profit can improve capacity allocation (Steinhardt and Gönsch, 2012) and overbooking (Alstrup *et al.*, 1986; Karaesmen and van Ryzin, 2004) in terms of revenue performance.

From a methodological point of view, the work by Chen *et al.* (2010) is related to our work. Unlike previous research, they conducted a structural analysis of optimal solutions to a dynamic seat-allocation model with two substitutable flights. The authors found that dynamic value functions satisfy a set of structural properties required to characterize the structure of optimal dynamic booking decisions. Since Lautenbacher and Stidham's (1999) classification of seat-allocation models in terms of the underlying demand model, booking decisions in dynamic models are understood as time-driven decisions because they primarily depend on the particular arrival times of customer requests. Dynamic models imply time-driven decisions due to the assumption that the booking horizon can be divided into sufficiently small booking periods such that at most one request occurs at a time, while different products may be requested concurrently. In static models, by contrast, booking decisions limit the number of customers to accept for each product and may therefore be termed demand-driven. Static seat-allocation models assume a predetermined arrival order in which the total demand for a particular product realizes in a designated period. Recently, static and dynamic models with a single leg have been analyzed in the context of overbooking (Aydın *et al.*, 2013). For single-leg, multi-fare seat-allocation models, Lautenbacher and Stidham (1999) first revealed that optimal booking decisions have a common underlying structure despite the different demand models. Their unifying model, which was recently employed in the structural analysis by Aydın *et al.* (2009), is known as the omnibus model.

Our research pursues a similar line of argument: We analyze the structure of optimal booking decisions from the perspective of an extended omnibus model with two substitutable resources and multiple flexible products. Our contributions to existing RM literature lie, firstly, in proving that static value functions propagate concavity, submodularity, and subconcavity just as dynamic value functions have been shown to do (Chen *et al.*, 2010). The results presented in the paper at hand, however, are based on less restrictive assumptions: We show that componentwise concavity and submodularity are sufficient to derive optimal booking policies for both dynamic and static value functions. Moreover, we propose a general optimal policy based on four monotonic switching curves. Secondly, we analyze the computational complexity of computing value functions and optimal booking decisions, and we herewith show that the respective dynamic programs can be solved efficiently.

The remainder of this paper is organized as follows: In Section 2, we introduce a generalized dynamic-programming formulation for the case with substitutable resources. Section 3 addresses the verification of structural properties that are required to establish our optimal policy. In Section 4, we derive a closed-form solution based on four monotonic switching curves solving static and dynamic models exactly. The computational complexity results are presented in Section 5, and final conclusions are drawn in Section 6.

2. Unifying problem formulation

This section is devoted to set up a unifying model that generalizes Lautenbacher and Stidham's (1999) omnibus model. The original omnibus model was introduced in order to marry static and dynamic seat allocation models for the multi-fare, single-leg case. We extend the omnibus model to capture products with multiple substitutable resources when accept/reject and assign decisions have to be made online.

Suppose an airline operates $m = |\mathcal{I}|$ flight legs and provides an assortment consisting of $n = |\mathcal{J}|$ specific and flexible products. A product is identified by its type $k \in \{1, \dots, K\}$ and fare class $j \in \{1, \dots, f_k\}$. We refer to a particular product by a pair (k, j). For notational ease, an identical number of fare classes $f_k = f$ is assumed for all types k. Thus, the set of products is denoted by $\mathcal{J} = \{1, \dots, K\} \times \{1, \dots, f\}$. Furthermore, a group of $d \in \mathbb{Z}_{\geq 0}$ requests for product (k, j) may be observed in any booking period $t \in \mathcal{T} = \{1, \dots, \tau\}$. For the event that no request occurs at a given time, we introduce a dummy product $(0, 0)$ and set $\mathcal{J}^0 = \mathcal{J} \cup \{(0, 0)\}$.

A product type k defines the product-specific subset of constituting resources $\mathcal{I}_k \subseteq \mathcal{I} = \{1, \dots, m\}$. Type k is specific if $|\mathcal{I}_k| = 1$, and flexible if $|\mathcal{I}_k| \geq 2$. The airline collects net revenue r^i_{kj} when a request for product (k, j) is booked on leg i, i.e., accepted and assigned to resource i. Thus, we have m-vectors $r_{kj} = (r^i_{kj})_{i \in \mathcal{I}}$ for each product $(k, j) \in \mathcal{J}$. It is conventional to assume that, for each product type, fare classes j are numbered by decreasing revenue so that $r_{k1} > \cdots > r_{kf}$ holds for each k.

We refer to the state at any given time as the (system) *booking level* $x = (x_i)_{i \in \mathcal{I}}$ with x_i being the number of reservations currently accepted and assigned to resource i. Let $c = (c_i)_{i \in \mathcal{I}}$ denote the initial system capacity. The system starts empty at time $t = \tau$ and booking is impossible after $t = 1$. The set $\mathcal{X}_t = \{x \in \mathbb{Z}^m : x \geq 0\}$ describes possible states for each booking period $t \in \mathcal{T}$. Note that $c - x$ is the available seat inventory when entering period t.

The number of product (k, j) requests accepted and assigned to resource i in period t is denoted by u_i. Thus, we model decisions as m-vectors $u_t = u_t(x, k, j, d) = (u_i)_{i \in \mathcal{I}}$. Each product-defining resource set is represented by a binary incidence m-vector $A_{kj} = (a^i_{kj})_{i \in \mathcal{I}}$, i.e., if a request for product (k, j) can be satisfied by employing resource i, a^i_{kj} has value one (zero otherwise). Note that we keep the standard *partial fulfillment* assumption. Let $C = \max_{i \in \mathcal{I}} c_i$ denote the largest available capacity and let $M \gg C$ be a large number. Then, $\mathcal{U}_t = \mathcal{U}_t(x, k, j, d) = \{u \in \mathbb{Z}^m : 0 \leq u \leq M A_{kj} \text{ and } \sum_{i=1}^m u_i \leq d\}$ describes the set of feasible actions for each period $t \in \mathcal{T}$ given that the current booking level is x and that d requests for product (k, j) are observed in period t.

We model the independent demand distribution by introducing probabilities p_{kjdt} for the event that $d \geq 1$ units of product (k, j) are requested at time t. Static and dynamic demand distributions are characterized by

$$\sum_{(k,j) \in \mathcal{J}^0} \sum_{d=0}^{\infty} p_{kjdt} = 1$$

for all $t \in \mathcal{T}$. For every $t \in \mathcal{T}$, it follows that the probability of no demand for product (k, j) is $p_{kj0t} = 1 - \sum_{d=1}^{\infty} p_{kjdt}$ and the probability of no demand in t is $p_{001t} = 1 - \sum_{(k,j) \in \mathcal{J}} p_{kj1t}$. In particular, let $(k(t), j(t))$ denote the product scheduled to arrive at time t. A distribution that satisfies, for all $t \in \mathcal{T}$,

$$\sum_{d=0}^{\infty} p_{k(t)j(t)dt} = 1 \quad \text{and} \quad p_{kjdt} = 0 \quad \text{if } k \neq k(t) \text{ or } j \neq j(t) \tag{1}$$

is called static demand distribution or *batch-request* model. The assumption that requests for distinct products arrive in non-overlapping booking periods leads to an exact pairing $\mathcal{T} \leftrightarrow \mathcal{J}, t \leftrightarrow (k(t), j(t))$ and thus $\tau = n$. For ease of presentation, we suppress the explicit dependence of a product on its scheduled arrival time as long as it is clear from the context, i.e., $(k, j) = (k(t), j(t))$.

On the other hand, requests for different products may overlap in some periods and it is assumed that at most one customer arrives at a time. A distribution that satisfies, for all $t \in \mathcal{T}$,

$$\sum_{(k,j)\in\mathcal{J}^0} p_{kj1t} = 1 \quad \text{and} \quad p_{kjdt} = 0 \quad \text{if } d \geq 2 \tag{2}$$

is called dynamic demand distribution or *single-request* model.

Note that assuming dynamic demand distributions (as in, e.g., Lee and Hersh, 1993; Lautenbacher and Stidham, 1999; Subramanian *et al.*, 1999) requires to dynamically forecast product-specific arrival probabilities. Total demand for each product becomes implicit in the number of booking periods. Using static demand distributions (e.g., Curry, 1990; Wollmer, 1992; Brumelle and McGill, 1993; Robinson, 1995; Li and Oum, 2002) requires, in contrast, to forecast total demand for each product explicitly.

Let the value function $V_t(x)$ denote the maximum expected revenue from periods $t, t-1, \ldots, 0$ when there are x reservations on hand. Then, the generalized omnibus model is given by

$$V_t(x) = \sum_{(k,j)\in\mathcal{J}^0} \sum_{d=0}^{\infty} p_{kjdt} \max_{u_t \in \mathcal{U}_t(x,k,j,d)} \left\{ r_{kj}^\top u_t + V_{t-1}(x + u_t) \right\} \quad \forall x \in \mathcal{X}_t, t \in \mathcal{T} \tag{3}$$

$$V_0(x) = \bar{r} \sum_{i=1}^{m} \min\{0, c_i - x_i\} \quad \forall x \in \mathcal{X}_0, \tag{4}$$

where $\bar{r}$ is the denied-boarding penalty. It is assumed greater than all fares, i.e., $\bar{r} > \max_{k,j,i}\{r_{kj}^i\}$. In this case, the boundary conditions (4) ensure that overbooking the capacity of any resource is never optimal. Alternatively, setting $V_0(x) = 0$ for all x and imposing additional constraints $x \leq c$ and $0 \leq u \leq c - x$ yields the same effect.

For the remainder of this paper, we will consider the two-leg case of model (3)-(4) only, i.e., $\mathcal{I} = \{1, 2\}$ and $m = 2$. Note that the number of product types K cannot be greater than the number of distinct resource sets so that $m = 2$ results in $K = 3$ (two specific types and one flexible type). However, our model still captures arbitrarily many products of a given type k because the number of fare classes f is not limited.

Let the first and second product type be specific using the first and second leg, respectively. The third type is flexible, i.e.,

$$(a_{kj}^i) = \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \in \mathbb{Z}^{2\times 3f}.$$

For the batch-request model (1), we can drop the first summation in (3) such that

$$\begin{aligned} V_t(x) = {} & p_{kj0t} V_{t-1}(x) \\ & + \begin{cases} \sum\limits_{d=1}^{\infty} p_{1jdt} \max\limits_{\substack{u_t \in \mathcal{U}_t(x,1,j,d): \\ \|u_t\| = 0,1,\ldots,d}} \{ r_{1j}^1 \|u_t\| + V_{t-1}(x + \|u_t\| e_1) \} & \text{if } k = 1 \\ \sum\limits_{d=1}^{\infty} p_{2jdt} \max\limits_{\substack{u_t \in \mathcal{U}_t(x,2,j,d): \\ \|u_t\| = 0,1,\ldots,d}} \{ r_{2j}^2 \|u_t\| + V_{t-1}(x + \|u_t\| e_2) \} & \text{if } k = 2 \\ \sum\limits_{d=1}^{\infty} p_{3jdt} \max\limits_{u_t \in \mathcal{U}_t(x,3,j,d)} \{ r_{3j}^\top u_t + V_{t-1}(x + u_t) \} & \text{if } k = 3 \end{cases} \end{aligned} \tag{RMFP-s}$$

for all $x = (x_1, x_2)^\top \in \mathcal{X}_t$ and $t \in \mathcal{T}$, where $\|x\| = \sum_{i=1}^n x_i$ denotes the unit norm and e_i is the i-th unit vector. We refer to RMFP-s as the *static value function.*

For the single-request model (2), omitting the second summation in (3) yields, for all $x = (x_1, x_2)^\top \in \mathcal{X}_t$ and $t \in \mathcal{T}$,

$$\begin{aligned} V_t(x) = {} & p_{001t} V_{t-1}(x) \\ & + \sum_{j=1}^{f} p_{1jt} \max\left\{r_{1j}^1 + V_{t-1}(x+e_1), V_{t-1}(x)\right\} \\ & + \sum_{j=1}^{f} p_{2jt} \max\left\{r_{2j}^2 + V_{t-1}(x+e_2), V_{t-1}(x)\right\} \\ & + \sum_{j=1}^{f} p_{3jt} \max\left\{r_{3j}^1 + V_{t-1}(x+e_1), r_{3j}^2 + V_{t-1}(x+e_2), V_{t-1}(x)\right\}. \end{aligned} \tag{RMFP-d}$$

We refer to RMFP-d as the *dynamic value function.* The terminal conditions are initialized by (4) in both cases. We emphasize that RMFP-d closely resembles the dynamic two-flight seat-allocation model used by Chen *et al.* (2010), except that in their model the price of the flexible product is always identical to the price of the corresponding specific products and the purchaser is assumed to be indifferent to the possible allocations. Beyond that scenario, our formulation allows for a pricing of flexible products that is independent of the fares of the specific products, e.g., flexible products may be sold at a discount in order to attract price sensitive customers.

3. Structural requirements

Our analysis relies on particular structural properties of the value function. Generally speaking, these properties guarantee global monotonicity of the value function with respect to changes of the two-dimensional integer state vector. To formally state the requirements, we adapt existing definitions (Zhuang and Li, 2010; Morton, 2006) to the forward difference operator $\Delta_i f(x) = f(x) - f(x + e_i)$, where $e_i \in \mathbb{Z}^m$ is the ith unit vector.

Definition 1. *Let $f : \mathbb{Z}^2 \to \mathbb{R}$ be a real-valued function. Then,*

(i) f is *(componentwise) concave* in i if $\Delta_i\Delta_i f(x) \le 0, \forall x \in \mathbb{Z}^2$ for some $i \in \{1, 2\}$
(ii) f is *submodular* in i and j if $\Delta_i\Delta_j f(x) \le 0, \forall x \in \mathbb{Z}^2$ for some $i, j \in \{1, 2\}, i \neq j$
(iii) f is *subconcave* in i and j if $\Delta_i\Delta_i f(x) \le \Delta_i\Delta_j f(x), \forall x \in \mathbb{Z}^2$ for some $i, j \in \{1, 2\}, i \neq j$.

Each of the properties (i)-(iii) describes a particular monotonic behaviour of the forward difference operators between adjacent points in a two-dimensional integer lattice whose coordinates are either horizontally, vertically, or in both dimensions offset by one. Note also that the properties in Definition 1 are defined with respect to a particular dimension or a a pair of dimensions. If a function is, for instance, subconcave in all distinct pairs of dimensions, it is said to be subconcave. Zhuang and Li (2010, 2012) introduced the concept of multimodularity to enhance the analysis of monotone optimal controls for a class of stochastic dynamic control problems. Multimodularity provides a toolbox of structural properties along with efficient proving techniques that help to make structural proofs more concise and to reduce the overall effort for writing down a formal proof. Our definition of multimodularity follows Zhuang and Li (2010):

Definition 2. *Let $f : \mathbb{Z}^2 \to \mathbb{R}$ be any real-valued function. f is* multimodular *if f satisfies properties (ii) and (iii) for all $x \in \mathbb{Z}^2$ and $i, j \in \{1, 2\}, i \neq j$.*

Morton (2006) provides a conceptually similar framework of structural properties that he calls directional modularity. He shows that submodulartity and subconcavity imply componentwise concavity. As a consequence, using multimodularity saves an explicit proof of componentwise concavity.

In the remainder of this section, we discuss whether multimodularity applies to the static and dynamic value functions. For this purpose, various transformations of the value function will be considered throughout our analysis. To keep the notation as clear as possible, we introduce two classes of control operators. Operators belonging to the first class are of the form $T_{kj}(W)(\cdot)$ defining some transformation of a function W. The subscript kj denotes that these transformations depend on product-specific parameters. The second class contains operators of the form $\widetilde{T}(W, q)(\cdot)$ that capture linear combinations of various transformations of the first class. Each such linear combination uses a given set of nonnegative weights that depends on some parameter q (time, usually).

3.1. Multimodularity of model RMFP-d

The main result of this subsection is the verification of multimodularity with regard to model RMFP-d. Thanks to existing structural results, we can simply state our result here without giving a rigorous proof of submodularity and subconcavity (see Lemma 1 in Chen *et al.* (2010); for their proof of properties (i)-(iii), however, the authors refer to Lin's (2008) dissertation, which is not freely available).

Our justification of proving a structurally analogous result is twofold: First, we achieve the result with less overall effort because we utilize the concept of multimodularity. Second, since some results in the next subsection rely on the structural properties of the dynamic value function, a thorough treatment of these seems to be essential.

To establish multimodularity of the dynamic value function, we define a maximization operator $T_{kj}(W)(\cdot)$ with $W = V_t(x)$, i.e.,

$$T_{kj}(V_t)(x) = \begin{cases} \max\left\{r^1_{1j} + V_t(x+e_1), V_t(x)\right\} & \text{if} \quad k=1 \\ \max\left\{r^2_{2j} + V_t(x+e_2), V_t(x)\right\} & \text{if} \quad k=2 \\ \max\left\{r^1_{3j} + V_t(x+e_1), r^2_{3j} + V_t(x+e_2), V_t(x)\right\} & \text{if} \quad k=3 \end{cases} \tag{5}$$

and a trivial operator $T_{00}(V_t)(x) = V_t(x)$ for the no-request event $(k, j) = (0, 0)$. The above transformations describe the problem of selling one unit of product (k, j) in period $t+1$ given $V_t(x)$ as a measure of the *revenue-to-go* of the current residual capacity $c - x$ from time t onwards. More specifically, T_{1j} and T_{2j} decide on the acceptance of a specific type 1 and type 2 request, respectively. In the event of a flexible request, T_{3j} instantaneously chooses to book that request on flight 1 or on flight 2, or to reject it.

Lemma 1. *If $V_t(x)$ is multimodular, then $T_{kj}(V_t)(x)$ is also multimodular for all $(k, j) \in \mathcal{J}^0$.*

Proof. See Section A in the Appendix. □

At the heart of Lemma 1 lies that each transformation performed by the maximization operator in (5) preserves the structural properties induced by the value function. Moreover, the next lemma states the well-known fact that elementary operations such as adding two functions and multiplying a function by a constant weight retain the induced properties.

Lemma 2. *Given a set of nonnegative weights $\alpha_t = \{\alpha_{kjt} \in \mathbb{R}_{\geq 0} : (k, j) \in \mathcal{J}^0\}$, let*

$$\widetilde{T}(V_t, t+1)(x) = \sum_{(k,j)\in\mathcal{J}^0} \alpha_{kj,t+1}\, T_{kj}(V_t)(x).$$

If $V_t(x)$ is multimodular, then $\widetilde{T}(V_t, t+1)(x)$ is also multimodular.

Proof. We refer the reader to Lautenbacher and Stidham (1999) and omit the proof. □

Note that the nonnegative weights are described by the dynamic probability distribution given in (2). Hence, taking expectation with respect to the arrival probabilities preserves multimodularity. The final result of this subsection is:

Lemma 3. *For model RMFP-d, the following result holds: If $V_0(x)$ is multimodular, then $V_t(x)$ is multimodular for all $t = 1, \dots, \tau$.*

Proof. The proof is essentially the same as Chen *et al.*'s (2010) proof of their Lemma 2. We reprove the result for completeness. The proof is by induction on t. The starting condition follows directly from the terminal conditions (4) because $\Delta_i V_0(x) = 0$ if $x_i < c_i$, $\Delta_i V_0(x) = \bar{r}$ if $x_i \geq c_i$ for all $i \in \{1, 2\}$, and $V_0(x + e_1) - V_0(x + e_2) = V_0(x + e_2) - V_0(x + e_1) = 0$ for all x. Repeated application of Lemmata 1 and 2 to the equation $V_t(x) = \widetilde{T}(V_{t-1}, t)(x)$ for all $t \in \mathcal{T}$ yields the result. □

Lemma 3 states that the dynamic value function propagates multimodularity with respect to time. Therefore, model RMFP-d particularly meets the structural requirements as given by Definition 1.

3.2. Multimodularity of model RMFP-s

The main result of this subsection is the verification of multimodularity with regard to RMFP-s. A major complicating issue stems from Definition 1, which assumes state transitions that are *bounded* by unit vectors. This assumption is obviously valid for single-request models. In batch-request models, by contrast, we are allowed to accept any integer number of customers at a time and to assign them to the available resources. Thus, state transitions can be represented by sums of integer multiples of the unit vectors. For d units of demand for a flexible product $(3, j)$ and for a given state x, the set of possible resulting states is

$$\left\{x + u \in \mathbb{Z}^2 : u = u_1 e_1 + u_2 e_2, \|u\| \leq d\right\} \qquad \text{for all demand levels } d \in \mathbb{Z}_{\geq 0}.$$

One approach would be to find a proper redefinition of the properties (i)-(iii) that covers the numerous possibilities to zigzag over the two-dimensional state space. But even if we had such a definition, the set of feasible states from which a particular control operator picks a best one can consist of far more than two or three elements. In general, the size of such a set can lead to a prohibitively large number of cases that need to be checked when proving structural properties.

We tackle this issue by reformulating the static value function in a way that transitions appear as if they were bounded by unit vectors. In batch-request models, demand for a particular product is observed as an aggregate quantity. That is, the task is to make an accept-and-assign decision for the entire group of customers at the very instant. Motivated by the partial fulfillment assumption, we suggest to process a group request customerwise. Therefore, we define a new recursive operator

$$H_{kj}(V_t, d)(x) = \begin{cases} V_t(x) & \text{if } d = 0 \\ \max\left\{r^1_{1j} + H_{1j}(V_t, d-1)(x + e_1), H_{1j}(V_t, d-1)(x)\right\} & \text{if } k = 1, d \geq 1 \\ \max\left\{r^2_{2j} + H_{2j}(V_t, d-1)(x + e_2), H_{2j}(V_t, d-1)(x)\right\} & \text{if } k = 2, d \geq 1 \\ \max\left\{r^1_{3j} + H_{3j}(V_t, d-1)(x + e_1), \right. & \\ \qquad \left. r^2_{3j} + H_{3j}(V_t, d-1)(x + e_2), H_{3j}(V_t, d-1)(x)\right\} & \text{if } k = 3, d \geq 1 \end{cases} \tag{6}$$

for all $(k, j) \in \mathcal{J}$. The key idea of the above recursion is to divide the original, unbounded batch-request problem into a series of nested, bounded single-request problems. The next lemma formalizes this idea.

Lemma 4. *Given a feasible state $x \in \mathcal{X}_t$ at time t, the following result holds:*

$$H_{kj}(V_t, d)(x) = \max_{u_t \in \mathcal{U}_t(x,k,j,d)} \{r^\top_{kj} u_t + V_t(x + u_t)\}$$

for all demand levels $d \in \{1, 2, \dots\}$ of product $(k, j) \in \mathcal{J}$.

Proof. We prove the result for operator $H_{3j}(V_t, d)(x)$ by induction over the demand levels. The base case $d = 1$ is true by definition, i.e., $H_{3j}(V_t, 1)(x) = \max\{r^1_{3j} + H_{3j}(V_t, 0)(x + e_1), r^2_{3j} + H_{3j}(V_t, 0)(x + e_2), H_{3j}(V_t, 0)(x)\} = \max\{r^1_{3j} + V_t(x + e_1), r^2_{3j} + V_t(x + e_2), V_t(x)\}$. We assume that the result holds for demand level $d - 1$.

Then, we have

$$
\begin{aligned}
H_{3j}(V_t,d)(x) &= \max\Big\{r^1_{3j} + H_{3j}(V_t,d-1)(x+e_1), r^2_{3j} + H_{3j}(V_t,d-1)(x+e_2), H_{3j}(V_t,d-1)(x)\Big\}\\
&= \max\Big\{r^1_{3j} + \max_{u_t\in\mathcal{U}_t(x+e_1,3,j,d-1)}\big\{r^\top_{3j}u_t + V_t(x+u_t+e_1)\big\},\\
&\qquad r^2_{3j} + \max_{u_t\in\mathcal{U}_t(x+e_2,3,j,d-1)}\big\{r^\top_{3j}u_t + V_t(x+u_t+e_2)\big\}, \max_{u_t\in\mathcal{U}_t(x,3,j,d-1)}\big\{r^\top_{3j}u_t + V_t(x+u_t)\big\}\Big\}\\
&= \max\Big\{\max_{u_t\in\mathcal{U}_t(x+e_1,3,j,d-1)}\big\{r^\top_{3j}(u_t+e_1) + V_t(x+u_t+e_1)\big\},\\
&\qquad \max_{u_t\in\mathcal{U}_t(x+e_2,3,j,d-1)}\big\{r^\top_{3j}(u_t+e_2) + V_t(x+u_t+e_2)\big\}, \max_{u_t\in\mathcal{U}_t(x,3,j,d-1)}\big\{r^\top_{3j}u_t + V_t(x+u_t)\big\}\Big\}\\
&= \max_{u_t\in\mathcal{U}_t(x,3,j,d)}\big\{r^\top_{3j}u_t + V_t(x+u_t)\big\}.
\end{aligned}
$$

The first equality is the definition, the second equality follows from the induction assumption, and the third equality follows from simple rearrangement. The last equality results from

$$\mathcal{U}_t(x,3,j,d) = \mathcal{U}_t(x+e_1,3,j,d-1) \cup \mathcal{U}_t(x+e_2,3,j,d-1) \cup \mathcal{U}_t(x,3,j,d-1),$$

which is also visualized in Figure 1.

As to products $(1,j)$ and $(2,j)$, note that the equalities

$$
\begin{aligned}
H_{1j}(V_t,d)(x) &= \max_{\substack{u_t\in\mathcal{U}_t(x,1,j,d):\\ \|u_t\|=0,1,\ldots,d}} \{r^1_{1j}\,\|u_t\| + V_t(x+\|u_t\|\,e_1)\}\\
H_{2j}(V_t,d)(x) &= \max_{\substack{u_t\in\mathcal{U}_t(x,2,j,d):\\ \|u_t\|=0,1,\ldots,d}} \{r^2_{2j}\,\|u_t\| + V_t(x+\|u_t\|\,e_2)\}
\end{aligned}
$$

are just the one-dimensional special cases of the above argument. Hence, we skip to show them. This completes the proof. □

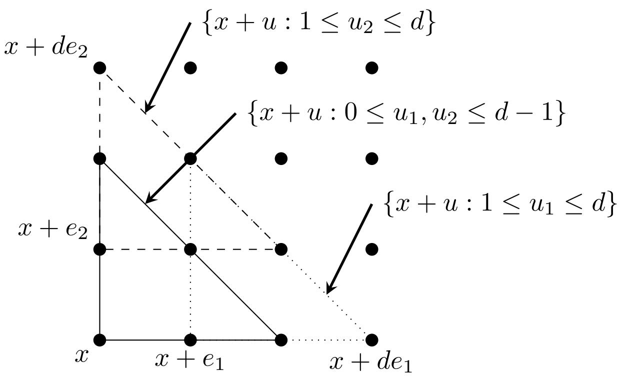

Figure 1: A cover of the transition set for a given root x

This result facilitates to reformulate model (RMFP-s) using the recursion 6.

Lemma 5. *The Bellman equations given by model RMFP-s are equivalent to*

$$V_t(x) = \begin{cases} \sum_{d=0}^{\infty} p_{1jdt}\, H_{1j}(V_{t-1}, d)(x) & \text{if } k = 1 \\ \sum_{d=0}^{\infty} p_{2jdt}\, H_{2j}(V_{t-1}, d)(x) & \text{if } k = 2 \\ \sum_{d=0}^{\infty} p_{3jdt}\, H_{3j}(V_{t-1}, d)(x) & \text{if } k = 3 \end{cases}$$

for all $x \in \mathcal{X}_t$ *and* $t \in \mathcal{T}$.

Proof. The result follows immediately by substituting $H_{kj}(V_t, d)(x)$ into the right hand side of the Bellman equations given by RMFP-s and by applying Lemma 4 for all $t \in \mathcal{T}$. □

This reformulation crucially simplifies our subsequent analysis which proceeds in two major steps: First, we show that the static value function propagates multimodularity with respect to the observed demand level at any given stage. Second, we show that the static value function propagates multimodularity with respect to time.

Lemma 6. *Suppose a time period* $t \in \mathcal{T}$ *is given. If* $V_t(x)$ *is multimodular, then* $H_{kj}(V_t, d)(x)$ *is multimodular for all* $d \in \{0, 1, \dots\}$ *and* $(k, j) \in \mathcal{J}$.

Proof. For $d = 0$, the result is trivial by definition of $H_{kj}(V_t, 0)(x) = V_t(x)$. For demand levels $d \in \{1, 2, \dots\}$, we show the result by induction on d. It is easy to see the result for $d = 1$ by noting the identities $T_{kj}(V_t)(x) = H_{kj}(V_t, 1)(x)$ for all $(k, j) \in \mathcal{J}$ and by applying Lemma 1. Assume that the result holds for demand level $d - 1$. Consider operator $T_{kj}(W)(x)$ given in (5) and let $W = H_{kj}(V_t, d - 1)$. Note the identities $H_{kj}(V_t, d)(x) = T_{kj}(H_{kj}(V_t, d - 1))(x)$ following from the recursive definition in (6) for each product $(k, j) \in \mathcal{J}$ and demand level $d \in \{1, 2, \dots\}$. Using these identities and Lemma 1, the result follows directly for all $d \in \{2, 3, \dots\}$. □

Next, we consider linear combinations of the above operators.

Lemma 7. *Given a set of nonnegative weights* $\{\alpha_{kjdt} \in \mathbb{R}_{\geq 0} : (k, j) \in \mathcal{J}, d \in \{0, 1, \dots\}\}$, *let*

$$\widetilde{H}(V_t, t+1)(x) = \begin{cases} \sum_{d=0}^{\infty} \alpha_{1jd,t+1}\, H_{1j}(V_t, d)(x) & \text{if } k = 1 \\ \sum_{d=0}^{\infty} \alpha_{2jd,t+1}\, H_{2j}(V_t, d)(x) & \text{if } k = 2 \\ \sum_{d=0}^{\infty} \alpha_{3jd,t+1}\, H_{3j}(V_t, d)(x) & \text{if } k = 3. \end{cases}$$

If $V_t(x)$ *is multimodular, then* $\widetilde{H}(V_t, t+1)(x)$ *is multimodular.*

Proof. See the proof of Lemma 2. □ With weights described by the probabilities in (1), the above Lemma states that taking expectation over a static demand distribution also preserves multimodularity.

Now, we are ready to establish the final result of this subsection.

Lemma 8. *For model RMFP-s, the following result holds: If* $V_0(x)$ *is multimodular, then* $V_t(x)$ *is multimodular for all* $t \in \mathcal{T}$.

Proof. We prove the result by induction over the periods. For the last period, the result follows from the terminal conditions (4). By Lemma 5, we can use the identitiy $V_t(x) = \widetilde{H}(V_{t-1}, t)(x)$ and apply Lemmata 6 and 7 for all $t \in \mathcal{T}$ repeatedly. This yiels the result. □ Note that Lemma 8 is rather general in the sense that it does not depend on a specific arrival sequence. The reason is that we have shown the propagation of multimodularity with respect to the time index, but without using a particular mapping of products to periods.

Remark 1. In the proofs of Lemmata 1 - 3 and Lemmata 6 - 8, it is possible to replace multimodular with properties (i) and (ii) meaning that if $V_0(x)$ is concave and submodular then the same holds for $V_t(x)$ for all $t \in \mathcal{T}$ in both models RMFP-s and RMFP-d.

4. Optimal accept-and-assign policies for two-flight models

In this section we derive optimal accept-and-assign policies based on the structural results of the preceding section. We start with the description of the case with specific products followed by the case with flexible products.

4.1. Handling specific product requests

To control the allocation of requests for specific products, we define two functions

$$S_{1jt} : \{0, \ldots, c_2\} \to \{0, \ldots, c_1\}, x_2 \mapsto S_{1jt}(x_2) = \min\left\{\hat{x}_1 \in \{0, 1, \ldots, c_1\} : \Delta_1 V_t(\hat{x}_1, x_2) - r^1_{1j} \geq 0\right\}$$

and

$$S_{2jt} : \{0, \ldots, c_1\} \to \{0, \ldots, c_2\}, x_1 \mapsto S_{2jt}(x_1) = \min\left\{\hat{x}_2 \in \{0, 1, \ldots, c_2\} : \Delta_2 V_t(x_1, \hat{x}_2) - r^2_{2j} \geq 0\right\}$$

for any feasible state $x = (x_1, x_2)^\top \in \mathcal{X}_t$ at time $t \in \mathcal{T}$. When in the above definitions the set is empty, we define $S_{1jt}(x_2) = c_1$ and $S_{2jt}(x_1) = c_2$. The functions S_{1jt} and S_{2jt} are so-called *switching curves* mapping a resource level in one dimension to a resource level in the other dimension. The next lemma establishes the monotonicity of the two switching curves.

Lemma 9. *For all $t = 1, \ldots, \tau$, $S_{1jt}(x_2)$ is nonincreasing in x_2, and $S_{2jt}(x_1)$ is nonincreasing in x_1 for both models RMFP-d and RMFP-s.*

Proof. The technique for proving this and similar results is rather general. The chain of arguments is the following: If $V(x)$ is concave and submodular, i.e., properties (i) and (ii) hold, then, for all dimensions i, the forward difference operators $\Delta_i V(x)$ are nondecreasing in x_1 and x_2. Moreover, several types of transformations of these operators $\Delta_i V(x)$ are also nondecreasing. Transformations maintaining nondecreasingness are, e.g., constant shifts, nonnegative linear transformations, and the minimum operator over two or more functions. Let $g(x)$ denote such a transformation resulting in a nondecreasing function. Then, choosing a fixed direction i, for ease of notation say $i = 1$, the function $S : (x_2, \ldots, x_m) \mapsto S(x_2, \ldots, x_m) = \min\{\hat{x}_1 : (\hat{x}_1, x_2, \ldots, x_m) \in \mathcal{X}, g(\hat{x}_1, x_2, \ldots, x_m) \geq 0\}$ is non*in*creasing. (The proof is straightforward.)

Our Lemmata 3 and 8 establish the concavity and submodularity of $V_t(x)$. Moreover, the transformations $g(x)$ used to define S_{1jt} and S_{2jt} are simple shifts of the forward difference operators by $-r^1_{1j}$ and $-r^2_{2j}$, respectively. This completes the proof for RMFP-s. For model RMFP-d, the result was shown by Chen *et al.* (2010). □

The critical values produced by the switching curves can be interpeted as inventory-sensitive *threshold levels* meaning controls for the allocation of specifc products that depend on the current booking level. Lemma 9 states that the higher the booking level of one flight, the lower the threshold level of the other flight, or in other words, the more restrictive we should be in accepting additional specific requests. This is because the marginal seat revenues $\Delta_i V_t(x)$ anticipate diminishing future revenue opportunities from the flexible demand segments when the residual capacity of any of the two legs decreases. We next devise a booking policy based on S_{1jt} and S_{2jt} that characterizes optimal actions regarding specific products.

Theorem 1. *Given any feasible booking level $x \in \mathcal{X}_t$ and any number $d \in \{0, 1, \ldots, \infty\}$ of customers of a specific product type $k \in \{1, 2\}$ requesting class j seats in period t, let*

$$\ell_{1jt}(x) = (S_{1j,t-1}(x_2) - x_1)^+ \quad \text{and} \quad \ell_{2jt}(x) = (S_{2j,t-1}(x_1) - x_2)^+$$

denote the inventory-sensitive seat allocation for product (k, j) at time t. Then, the optimal decision functions in models RMFP-s and RMFP-d are given by

$$u^*_t(x, 1, j, d) = \min\{d, \ell_{1jt}(x)\}e_1 \quad \text{and} \quad u^*_t(x, 2, j, d) = \min\{d, \ell_{2jt}(x)\}e_2.$$

Proof. We show the result for RMFP-s. Let us consider any booking period in which customers of the first specifc product type are scheduled to arrive, i.e., $t \in \{t \in \mathcal{T} : k(t) = 1 \text{ and } j(t) = j \in \{1, \dots, f\}\}$. In this case, we can write:

$$\begin{aligned} V_t(x) &= \sum_{d=0}^{\infty} p_{1jdt} \max_{\substack{u_t \in \mathcal{U}_t(x,1,j,d): \\ \|u_t\|=0,1,\dots,d}} \left\{ r_{1j}^1 \|u_t\| + V_{t-1}(x + \|u_t\| e_1) \right\} \\ &= V_{t-1}(x) + \sum_{d=0}^{\infty} p_{1jdt} \max_{\substack{u_t \in \mathcal{U}_t(x,1,j,d): \\ \|u_t\|=0,1,\dots,d}} \left\{ \sum_{z=1}^{\|u_t\|} \left(r_{1j}^1 - \Delta V_{t-1}(x + ze_1 - e_1) \right) \right\} \\ &= V_{t-1}(x) + \sum_{d=0}^{\infty} \sum_{z=1}^{\min\{d, \ell_{1jt}(x)\}} p_{1jdt} \left(r_{1j}^1 - \Delta V_{t-1}(x + ze_1 - e_1) \right). \end{aligned}$$

Note that the first equality is the definition and the second equality follows from $V_t(x) - V_t(x + \|u_t\| e_1) = \sum_{z=1}^{\|u_t\|} \Delta_1 V_t(x + ze_1 - e_1)$, where we take the value of the summation to be zero if $d = 0$. The third equality follows from the definition of $S_{1j,t-1}$ and the nondecreasingness of $\Delta_1 V_t(x)$ in x_1, which imply, for all $x \in \mathcal{X}_t$ and for each realized demand level $z \in \{1, 2, \dots\}$,

$$\Delta_1 V_{t-1}(x + ze_1 - e_1) \begin{cases} < r_{1j}^1 & \text{if } z \le \ell_{1jt}(x) \Leftrightarrow x_1 + z \le S_{1j,t-1}(x_2) \\ \ge r_{1j}^1 & \text{otherwise.} \end{cases}$$

Hence, the revenue maximizing number of product $(1, j)$ customers to accept is equal to $\min\{d, \ell_{1jt}(x)\}$, as desired. By symmetry, the respective result follows for products $(2, j)$.

To obtain the result for RMFP-d, note that we only need to use the special case of the above argument with $z = 1$. This was proven earlier by Chen *et al.* (2010). □

Theorem 1 provides a closed-form policy to control the sale of specific products in RMFP-s and RMFP-d. For a given system booking level $\bar{x}$, Figure 2 visualizes the optimal acceptance region for type 1 requests on the left-hand side of $S_{1j,t-1}$ and the acceptance region of type 2 requests below $S_{2j,t-1}$. In line with single-leg RM theory, it is easy to see that optimal controls for the same specific product type $k \in \{1, 2\}$ are *nested by fare class*. For instance, consider the static case with customers arriving in low-before-high revenue order: For any two booking periods $t > t'$ which imply $r_{kj(t)} < r_{kj(t')}$ and any feasible booking level x_i on flight i not used by product type k, we have $S_{kj(t),t-1}(x_i) \le S_{kj(t'),t'-1}(x_i)$, that is, customers of product $(k, j(t'))$ are accepted if customers of product $(k, j(t))$ are accepted.

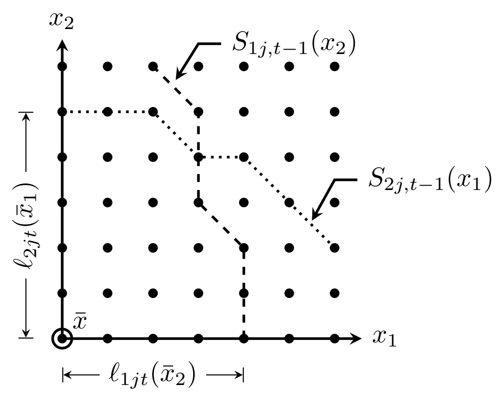

Figure 2: Optimal acceptance region and seat allocations for specific product types

4.2. Handling flexible product requests

The task of allocating seats to flexible customers can be viewed as simultaneously accepting/rejecting customers (product level) and, when accepted, assigning them to a flight (resource level). Chen *et al.* (2010) proposed a time-driven booking control for their dynamic model based on switching curves. For the omnibus model, i.e., both the static and dynamic case, we define the following two functions:

$$S_{3jt} : \{0, \dots, c_2\} \to \{0, \dots, c_1\}, x_2 \mapsto S_{3jt}(x_2) = \min\{\hat{x}_1 : \min\{\Delta_1 V_t(\hat{x}_1, x_2) - r^1_{3j}, \Delta_2 V_t(\hat{x}_1, x_2) - r^2_{3j}\} \geq 0\}$$

and

$$\bar{S}_{3jt} : \{0, \dots, c_1\} \to \{0, \dots, c_2\}, x_1 \mapsto \bar{S}_{3jt}(x_1) = \min\{\hat{x}_2 : \min\{\Delta_1 V_t(x_1, \hat{x}_2) - r^1_{3j}, \Delta_2 V_t(x_1, \hat{x}_2) - r^2_{3j}\} \geq 0\}.$$

As before, in cases where the respective sets are empty, we define $S_{3jt}(x_2) = c_1$ and $\bar{S}_{3jt}(x_1) = c_2$.

We will show that all states resulting from optimal decisions $u^*_t(\bar{x}, 3, j, d)$ are inside the union of the regions inscribed by the two switching curves $S_{3j,t-1}$ and $\bar{S}_{3j,t-1}$ and the half-space formed by points not smaller than the current state $\bar{x} \in \mathcal{X}_t$, see Figure 3. For notational ease, we define the optimal acceptance region

$$\mathcal{R}_{jt} = \{x = (x_1, x_2)^\top \in \mathcal{X}_t : x_1 \leq S_{3j,t-1}(x_2) \text{ or } x_2 \leq \bar{S}_{3j,t-1}(x_1)\}.$$

Lemma 10. *For models RMFP-s and RMFP-d, the following result holds: Let $\bar{x} \in \mathcal{R}_{jt}$ be a given state and $u^* = u^*_t(\bar{x}, 3, j, d)$ be an optimal decision provided that the demand d of product $(3, j)$ has been observed in period t. Then, the resulting state fulfills $\bar{x} + u^* \in \mathcal{R}_{jt}$.*

Proof. The case $\bar{x} + u^*$ has one or both component 0 is trivial. We therefore assume that the resulting state $x' = \bar{x} + u^*$ has both components > 0. Since u^* is an optimal decision, we have $\Delta_1 V_{t-1}(x' - e_1) < r^1_{3j}$ or $\Delta_2 V_{t-1}(x' - e_2) < r^2_{3j}$ (or both). In the first case and using that $x' - e_1 = (x'_1 - 1, x'_2)^\top$, the above result gives $\Delta_1 V_{t-1}(x' - e_1) - r^1_{3j} < 0$ and herewith $x'_1 - 1 < S_{3j,t-1}(x'_2)$, i.e., $x'_1 \leq S_{3j,t-1}(x'_2)$. Note that for the last implication we used that the forward difference operators are nondecreasing in both RMFP-s and RMFP-d. Similarly, the second case results in $x'_2 \leq \bar{S}_{3j,t-1}(x'_1)$, which completes the proof. □

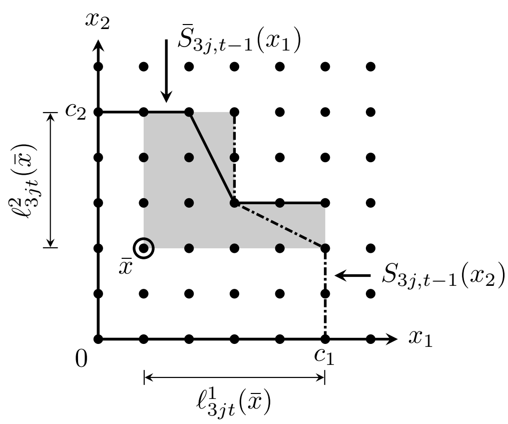

Figure 3: Optimal acceptance region for flexible products $(3, j)$

The next lemma establishes the monotonicity of the two functions S_{3jt} and $\bar{S}_{3jt}$.

Lemma 11. *For all $t = 1, \dots, \tau$, $S_{3jt}(x_2)$ is nonincreasing in x_2, and $\bar{S}_{3jt}(x_1)$ is nonincreasing in x_1 for models RMFP-d and RMFP-s.*

Proof. We can use the same arguments as in Lemma 9. Thus, we only need to consider the minimization term in the definition of $S_{3jt}(x_2)$, i.e., the function $g(x_1, x_2) = \min\{\Delta_1 V_t(x_1, x_2) - r^1_{3j}, \Delta_2 V_t(x_1, x_2) - r^2_{3j}\}$.

Due to the multimodularity of $V_t(x)$, the forward difference operators as well as functions resulting from constant shifts and the minimization operator are nondecreasing. Therefore, g is nondecreasing which completes the proof for S_{3jt}. The same reasoning proves the claim for $\bar{S}_{3jt}$. □

Defining the *resource level booking limit* as $\ell^1_{3jt}(x) = (S_{3j,t-1}(x_2)-x_1)^+$ and $\ell^2_{3jt}(x) = (\bar{S}_{3j,t-1}(x_1)-x_2)^+$, see Figure 3, Lemmata 10 and 11 have the following consequences:

1. An optimal decision u^* fulfills $\|u^*\| \leq \ell^1_{3jt}(\bar{x}) + \ell^2_{3jt}(\bar{x})$. The computation of this allotment requires to lookup the values $S_{3j,t-1}(\bar{x})$ and $\bar{S}_{3j,t-1}(\bar{x})$. Using binary search (Talluri and van Ryzin, 2004), each lookup requires $\mathcal{O}(\log C)$ or even less time in the worst case.
2. Under a naive strategy, the exact determination of $\mathcal{R}_{jt}$ requires C lookups in $S_{3j,t-1}$ and $\bar{S}_{3j,t-1}$, in the worst case. However, we can further reduce the number of required lookups from $\mathcal{O}(C \log C)$ to $\mathcal{O}(\log C) + \mathcal{O}(C) = \mathcal{O}(C)$ by exploiting the argument in Lemma 11, i.e., the monotonicity of the transformation g; we describe such a look-up procedure with regard to $S_{3j,t-1}$, see Algorithm 2 in Appendix B. Note that the same technique is applicable for computing S_{1jt} and S_{2jt}.

We can partially describe an optimal policy now:

Theorem 2. *For models RMFP-s and RMFP-d, the following result holds: Let $t \in \mathcal{T}$ be given. For any $x \in \mathcal{X}_t$, define the inventory-sensitive product level booking limit*

$$\ell_{3jt}(x) = \max_{x' \in \mathcal{R}_{jt}: x' \geq x} \|x' - x\| . \tag{7}$$

An optimal policy accepts exactly $\min\{d, \ell_{3jt}(\bar{x})\}$ requests of a flexible product $(3, j)$ for all observed demands $d \in \{0, 1, \ldots\}$. If $\mathcal{R}_{jt}$ is known, then $\ell_{3jt}(\bar{x})$ can be computed in $\mathcal{O}(C)$ time.

Example 1. We consider the situation depicted in Figure 3, i.e., the current state $\bar{x} = (c_1 - 4, c_2 - 3)$. We know that we will never accept more than $\ell^1_{3jt}(\bar{x}) + \ell^2_{3jt}(\bar{x}) = 4 + 3 = 7$ requests. This is an upper bound only, since $\ell_{3jt}(\bar{x}) = 5$. The optimal decision is to accept up to five requests of product $(3, j)$.

Suppose that we observe exactly $d = 5$ requests. The two possible resulting states are then $x' = (c_1 - 2, c_2)^\top$ and $x'' = (c_1, c_2 - 2)^\top$. In the first case, we assign two customers to leg 1 and three customers to leg 2, while in the second case four customers are assigned to leg 1 and one to leg 2. One still has to check which state is preferable.

If the observed demand were $d = 3$, there would be four possible allocations. Also in this case, one still has to find a best allocation. In this sense, we have derived only a partial description of the structure of an optimal policy so far.

Note that once acceptance decisions are made at some point in time based on the rule given by (7), the resulting state is guaranteed to be within the optimal acceptance region at any future point in time. To establish this result, consider the following lemma:

Lemma 12. *For models RMFP-s and RMFP-d, the value function is submodular with respect to time, i.e.,*

(i) $\Delta_1 V_t(x) \geq \Delta_1 V_{t-1}(x) \quad \forall x \in \mathcal{X}_t, t \in \mathcal{T}$
(ii) $\Delta_2 V_t(x) \geq \Delta_2 V_{t-1}(x) \quad \forall x \in \mathcal{X}_t, t \in \mathcal{T}$.

Proof. See Section A in the Appendix. □

The monotonicity result of this lemma is intuitive because the marginal seat revenues decrease as time approaches the end of the booking horizon. Using the definitions of S_{3jt} and $\bar{S}_{3jt}$ and Lemma 12 leads to the following direct consequences:

Theorem 3. *For models RMFP-s and RMFP-d, the following results hold:*

(i) *For any given state $(x_1, x_2)^\top \in \mathcal{X}_t$, $S_{3jt}(x_2)$ and $\bar{S}_{3jt}(x_1)$ are nonincreasing in t.*
(ii) *The optimal acceptance region is nonincreasing in t, i.e., $\mathcal{R}_{jt} \subseteq \mathcal{R}_{j,t-1}$ for all $t \in \mathcal{T}$.*

(iii) *Let* $\bar{x} \in \mathcal{R}_{jt}$ *be a given state and* $u^* = u_t^*(\bar{x}, 3, j, d)$ *be an optimal decision provided that the demand* d *of product* $(3, j)$ *has been observed in period* t. *Then,* $\bar{x} + u^* \in \mathcal{R}_{jt} \subseteq \mathcal{R}_{j,t-1} \subseteq \ldots \subseteq \mathcal{R}_1$.

As with the optimal booking limits for specific products, it is easy to show that an optimal product level booking limit for the flexible type $k = 3$ is also a nested control. In case of RMFP-d, we have

$$\ell_{31t}(x) \geq \ell_{32t}(x) \geq \ldots \geq \ell_{3ft}(x) \quad \text{for all } t \in \mathcal{T}.$$

5. Computational Complexity

Substituting the optimal booking limit controls for specific products described in Theorem 1 and those for flexible products described in Theorem 2 into the omnibus model results in the recursion

$$\begin{aligned} V_t(x) = & \sum_{\substack{(k,j) \in \mathcal{J}^0 : \\ k \in \{1,2\}}} \sum_{d=0}^{\infty} p_{kjdt} \big(r_{kj}^\top \min\{d, \ell_{kjt}(x)\} A_{kj} + V_{t-1}(x + \min\{d, \ell_{kjt}(x)\} A_{kj}) \big) \\ & + \sum_{(3,j) \in \mathcal{J}^0} \sum_{d=0}^{\infty} p_{3jdt} \max_{\substack{x + u_t \in \mathcal{R}_{jt} : \\ \|u_t\| = \min\{d, \ell_{3jt}(x)\}}} \{ r_{3j}^\top u_t + V_{t-1}(x + u_t) \} \end{aligned} \tag{8}$$

for all $x \in \mathcal{X}_t, t \in \mathcal{T}$.

The computational procedure to solve the above recursion is summarized in Algorithm 1. The indicated computational complexities follow from the results presented in Section 4. Lines 9 and 20-28 of Algorithm 1 present the only new aspect therein. For the dynamic and static case, respectively, these lines describe the computation of an optimal solution $u_t^* = u_t^*(x, k, j, d)$ to the problem of finding a revenue maximizing assignment when the number $\min\{d, \ell_{3jt}(x)\}$ of *accepted* flexible products $(3, j)$ is known for any given booking level $x \in \mathcal{X}_t$ at time t. Formally, this problem is delineated by the maximization on the right hand side of (8). In the following, we refer to a particular instance of the maximization problem as assignment problem. Some instances of the assignment problem were previously sketched by Example 1.

In RMFP-d, the complexity of solving an assignment problem indicated in line 9 is $\mathcal{O}(1)$ because exactly two possible contributions to $V_t(x)$ corresponding to the resulting states $x + e_1$ and $x + e_2$ have to be evaluated after we have looked up whether $\ell_{3jt}(x) \geq 1$. (The contribution is the respective summand in (8).) Therefore, the complexity of finding an optimal acceptance and assignment decision is dominated by the computation of $\mathcal{R}_{jt}$ requiring $\mathcal{O}(fC)$ time for each flexible product $(3, j)$.

In RMFP-s, we start with the computation of the contribution to $V_t(x)$ for all cases in which we accept exactly ℓ requests, i.e., when the demand d is not smaller than ℓ; see lines 20–22. For smaller demand realizations $d < \ell$, we use the fact that $u_t^*(x, k, j, d-1)$ is either $u_t^*(x, k, j, d) - e_1$ or $u_t^*(x, k, j, d) - e_2$. This is a direct consequence of the reformulation of the static value function $V_t(x)$ with the help of $H_{kj}(V_t, d)(x)$ as presented in Lemma 5. Therefore, we consider demand realizations in the ordering $d = \ell - 1, \ell - 1, \ldots, 2, 1$ in the loop in line 23. Note that a brute force strategy is to fully explore the optimal acceptance region in $\mathcal{O}(C^2)$ time. However, the above procedure reduces the computational burden to $\mathcal{O}(C)$.

The presented analysis of Algorithm 1 leads to the following theorem:

Theorem 4. *For the omnibus model given by* (8), *the following complexity results hold:*
(i) *RMFP-d can be solved in* $\mathcal{O}(\tau f C^2)$.
(ii) *RMFP-s can be solved in* $\mathcal{O}(f C^3)$.

The result of Theorem 4 can be seen as an extension of a well-known complexity result from single-leg RM theory (see Talluri and van Ryzin, 2004) stating that the solution of static and dynamic value functions both have the same computational complexity of $\mathcal{O}(fC^2)$ for the single-leg case. Here, for flexible products and two legs, the assumption $\tau \in \mathcal{O}(C)$ also yields an identical effort of $\mathcal{O}(\tau f C^2) = \mathcal{O}(fC^3)$.

By further exploiting the monotonicity of the dynamic value function, it is possible to derive a full characterization of optimal booking decisions using an additional switching curve for each product $(3, j)$ (see Chen *et al.*, 2010). However, their method is not able to gain any computational advantage over the procedure shown in Algorithm 1.

Algorithm 1: Solution procedure for the omnibus model

for $t = \tau, \tau - 1, \ldots, 2, 1$ **do** // dyn: $\mathcal{O}(\tau)$, stat: $\mathcal{O}(f)$
- Compute $S_{1j,t-1}, S_{2j,t-1}, \mathcal{R}_{jt}$ for all $(k, j) \in \mathcal{J}$ arriving in t // dyn: $\mathcal{O}(fC)$, stat: $\mathcal{O}(C)$
- **for** $x \in \mathcal{X}_t$ **do** // $\mathcal{O}(C^2)$
  - **case** *RMFP-d*
    - Set $V_t(x) := p_{001t} V_{t-1}(x)$ // $\mathcal{O}(1)$
    - **for** $(k, j) \in \mathcal{J}$ **do** // $\mathcal{O}(f)$
      - **case** $k = 1$ *or* $k = 2$
        - Compute $\ell_{kjt}(x)$ // $\mathcal{O}(1)$
      - **case** $k = 3$
        - Compute $\ell_{kjt}(x)$ and $u_t^* = u_t^*(x, k, j, 1)$ // $\mathcal{O}(1)$
      - Add contribution of $(k, j, 1, t)$ to $V_t(x)$ // $\mathcal{O}(1)$
  - **case** *RMFP-s*
    - Set $(k, j) := (k(t), j(t))$ // $\mathcal{O}(1)$
    - Set $V_t(x) := p_{kj0t} V_{t-1}(x)$ // $\mathcal{O}(1)$
    - **case** $k = 1$ *or* $k = 2$
      - Set $\ell := \ell_{kjt}(x)$ // $\mathcal{O}(C)$
      - **for** $d = 1, 2, \ldots, \ell - 1$ **do** // $\mathcal{O}(C)$
        - Add contribution of (k, j, d, t) to $V_t(x)$ // $\mathcal{O}(1)$
      - Add contribution of (k, j, d, t) for all $d \geq \ell$ to $V_t(x)$ // $\mathcal{O}(1)$
    - **case** $k = 3$
      - Set $\ell := \ell_{kjt}(x)$ // $\mathcal{O}(C)$
      - Compute $u_t^* = u_t^*(x, k, j, \ell)$ with $\|u_t^*\| = \ell$ // $\mathcal{O}(C)$
      - Add contribution of (k, j, d, t) for all $d \geq \ell$ to $V_t(x)$ // $\mathcal{O}(1)$
      - **for** $d = \ell - 1, \ell - 2, \ldots, 2, 1$ **do** // $\mathcal{O}(C)$
        - **if** $V_{t-1}(x + u_t^* - e_1) > V_{t-1}(x + u_t^* - e_2)$ **then** // $\mathcal{O}(1)$
          - Set $u_t^* := u_t^* - e_1$ // $\mathcal{O}(1)$
        - **else**
          - Set $u_t^* := u_t^* - e_2$ // $\mathcal{O}(1)$
        - Add contribution of (k, j, d, t) to $V_t(x)$ // $\mathcal{O}(1)$

6. Conclusions

In this paper, we extended Lautenbacher and Stidham's (1999) omnibus model to the case of multiple flexible products. Omnibus models unify static and dynamic seat allocation models. To date, static models with multiple resources, despite their theoretical and practical relevance, have not yet been structurally analyzed in the revenue management literature. One reason seems to be that customers arriving in batches lead to major complications in a theoretical analysis. For the analysis of models with two substitutable resources, we showed that three important structural properties propagate: These properties are concavity, submodularity, and subconcavity, together known as multimodularity. Propagation of properties means that if the boundary conditions of the Bellman equations satisfy them, the entire value function does also. For static value functions, we introduced a recursive reformulation that reduces the batch-request problem to a series of simpler single-request problems. In particular, it helps to simplify not only the theoretical analysis but also the computational approach. Due to the recursive reformulation, the propagation of multimodularity for the static case is proven for the first time.

Moreover, we derived optimal acceptance and assignment policies for the extended omnibus model so that optimal policies can be described for both the dynamic and the static case. While known results for the dynamic case rely on the multimodularity property, our policies need only concave and submodular value functions describing the boundary conditions of the Bellman equations. Thus, compared to the work by Chen *et al.* (2010) we need less assumptions and cover the more general omnibus model.

Finally, we analyzed the computational effort of implementing the policies. Compared to the single-leg case, the computational effort to solve the Bellman equations grows by factor C (the largest available capacity). The proposed method has an overall computational complexity of $\mathcal{O}(fC^3)$ (f is the number of fare classes) and is therefore efficient. We expect it to be of practical use as a component in revenue management systems.

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Appendix

A. Proofs

Lemma 1. *If $V_t(x)$ is multimodular, then $T_{kj}(V_t)(x)$ is also multimodular for all $(k,j) \in \mathcal{J}^0$.*

Proof. The operator $T_{00}(V_t)(x)$ is multimodular, trivially, because $V_t(x)$ is multimodular by assumption. We need to examine whether operator $T_{1j}(V_t)(x)$ is submodular and subconcave.

First, as to submodularity, it suffices to ascertain whether $\Delta_1 T_{1j}(V_t)(x)$ increases in x_2, since the reverse follows by symmetry. That is, we prove that $T_{1j}(V_t)(x) - T_{1j}(V_t)(x+e_1) \le T_{1j}(V_t)(x+e_2) - T_{1j}(V_t)(x+e_1+e_2)$, or equivalently,

$$\begin{aligned} &\max\{r^1_{1j} + V_t(x+e_1), V_t(x)\} - \max\{r^1_{1j} + V_t(x+2e_1), V_t(x+e_1)\} \\ &\quad \le \max\{r^1_{1j} + V_t(x+e_1+e_2), V_t(x+e_2)\} - \max\{r^1_{1j} + V_t(x+2e_1+e_2), V_t(x+e_1+e_2)\}. \end{aligned} \tag{9}$$

We apply the *case-by-case method* by Zhuang and Li (2010). As a consequence of their Lemma 2 (see Zhuang and Li, 2010, p. 464), inequality (9) can be established by analyzing only $2^2 = 4$ cases. In particular, we have

$$\begin{aligned} &V_t(x+e_1) - V_t(x+e_1) \\ &\quad = V_t(x+e_1+e_2) - V_t(x+e_1+e_2), \end{aligned} \tag{case 1}$$

$$\begin{aligned} &V_t(x+e_1) - V_t(x+2e_1) \\ &\quad \le V_t(x+e_1+e_2) - V_t(x+2e_1+e_2), \end{aligned} \tag{case 2}$$

$$\begin{aligned} &V_t(x) - V_t(x+e_1) \\ &\quad \le V_t(x+e_1) - V_t(x+2e_1) \\ &\quad \le V_t(x+e_1+e_2) - V_t(x+2e_1+e_2), \end{aligned} \tag{case 3}$$

$$\begin{aligned} &V_t(x) - V_t(x+e_1) \\ &\quad \le V_t(x+e_2) - V_t(x+e_1+e_2). \end{aligned} \tag{case 4}$$

Note that the second inequality in case 3 and the inequalities in case 2 and case 4 follow because V_t is $\Delta_2 V_t(x_1, x_2)$ is nondecreasing in x_2 by assumption. The first inequality in case 3 is true because $\Delta_1 V_t(x_1, x_2)$ is nondecreasing in x_1 by assumption.

Combining the results of case 1 and case 2, we can conclude

$$\begin{aligned} &r^1_{1j} + V_t(x+e_1) - \max\{r^1_{1j} + V_t(x+2e_1), V_t(x+e_1)\} \\ &\quad \le \max\{r^1_{1j} + V_t(x+e_1+e_2), V_t(x+e_2)\} - \max\{r^1_{1j} + V_t(x+2e_1+e_2), V_t(x+e_1+e_2)\}, \end{aligned} \tag{10}$$

and it also follows from case 3 and case 4

$$\begin{aligned} &V_t(x) - \max\{r^1_{1j} + V_t(x+2e_1), V_t(x+e_1)\} \\ &\quad \le \max\{r^1_{1j} + V_t(x+e_1+e_2), V_t(x+e_2)\} - \max\{r^1_{1j} + V_t(x+2e_1+e_2), V_t(x+e_1+e_2)\}. \end{aligned} \tag{11}$$

The partial results (10) and (11) together show that (9) holds. Hence, operator $T_{1j}(V_t)(x)$ is submodular.

Second, proceeding to subconcavity, it suffices, by symmetry, to show that $T_{1j}(V_t)(x+e_1) - T_{1j}(V_t)(x+e_2)$ is nonincreasing in x_1. That is, we prove that $T_{1j}(V_t)(x+e_1) - T_{1j}(V_t)(x+e_2) \ge T_{1j}(V_t)(x+2e_1) - T_{1j}(V_t)(x+e_1+e_2)$, or equivalently,

$$\begin{aligned} &\max\{r^1_{1j} + V_t(x+2e_1), V_t(x+e_1)\} - \max\{r^1_{1j} + V_t(x+e_1+e_2), V_t(x+e_2)\} \\ &\quad \ge \max\{r^1_{1j} + V_t(x+3e_1), V_t(x+2e_1)\} - \max\{r^1_{1j} + V_t(x+2e_1+e_2), V_t(x+e_1+e_2)\}. \end{aligned} \tag{12}$$

To verify inequality (12), we consider the following cases:

$$\begin{aligned} &V_t(x+2e_1) - V_t(x+e_1+e_2) \\ &\quad \geq V_t(x+3e_1) - V_t(x+2e_1+e_2), \end{aligned} \tag{case 1}$$

$$\begin{aligned} &V_t(x+2e_1) - V_t(x+e_1+e_2) \\ &\quad = V_t(x+2e_1) - V_t(x+e_1+e_2), \end{aligned} \tag{case 2}$$

$$\begin{aligned} &V_t(x+e_1) - V_t(x+e_2) \\ &\quad \geq V_t(x+2e_1) - V_t(x+e_1+e_2) \\ &\quad \geq V_t(x+3e_1) - V_t(x+2e_1+e_2). \end{aligned} \tag{case 3+4}$$

Note that the inequalitites of case 1 and case 3+4 hold because $V_t(x+e_1) - V_t(x+e_2)$ is nonincreasing in x_1 by assumption. Now, from case 1 and case 3+4, we can follow

$$\begin{aligned} &\max\{r^1_{1j} + V_t(x+2e_1), V_t(x+e_1)\} - r^1_{1j} + V_t(x+e_1+e_2) \\ &\quad \geq \max\{r^1_{1j} + V_t(x+3e_1), V_t(x+2e_1)\} - \max\{r^1_{1j} + V_t(x+2e_1+e_2), V_t(x+e_1+e_2)\} \end{aligned}$$

and from case 3+4, it follows

$$\begin{aligned} &\max\{r^1_{1j} + V_t(x+2e_1), V_t(x+e_1)\} - V_t(x+e_2) \\ &\quad \geq \max\{r^1_{1j} + V_t(x+3e_1), V_t(x+2e_1)\} - \max\{r^1_{1j} + V_t(x+2e_1+e_2), V_t(x+e_1+e_2)\} \end{aligned}$$

Consequently, operator $T_{1j}(V_t)$ is subconcave and is thus multimodular.

The proof for T_{2j} is skipped since the procedure is essentially the same as above.

Turning to operator T_{3j}, we want to verify, first, that $\Delta_1 T_{3j}(V_t)(x)$ is nondecreasing in x_2, or equivalently,

$$\begin{aligned} &\max\left\{r^1_{1j} + V_t(x+e_1), r^2_{2j} + V_t(x+e_2), V_t(x)\right\} \\ &\qquad - \max\left\{r^1_{1j} + V_t(x+2e_1), r^2_{2j} + V_t(x+e_1+e_2), V_t(x+e_1)\right\} \\ &\quad \leq \max\left\{r^1_{1j} + V_t(x+e_1+e_2), r^2_{2j} + V_t(x+2e_2), V_t(x+e_2)\right\} \\ &\qquad\qquad - \max\left\{r^1_{1j} + V_t(x+2e_1+e_2), r^2_{2j} + V_t(x+e_1+2e_2), V_t(x+e_1+e_2)\right\}. \end{aligned} \tag{13}$$

Applying the same case-by-case procedure to inequality (13), $3^2 = 9$ cases have to be checked. For the first 3 cases, since $\Delta_1 V_t(x)$ and $\Delta_2 V_t(x)$ are both nondecreasing in x_1 and in x_2 by assumption, we can state

$$\begin{aligned} &V_t(x+e_1) - V_t(x+2e_1) \\ &\quad \leq V_t(x+e_1+e_2) - V_t(x+2e_1+e_2), \\ &V_t(x+e_1) - V_t(x+e_1+e_2) \\ &\quad \leq V_t(x+e_1+e_2) - V_t(x+e_1+2e_2), \\ &V_t(x+e_1) - V_t(x+e_1) \\ &\quad = V_t(x+e_1+e_2) - V_t(x+e_1+e_2), \end{aligned}$$

which together yield the result

$$\begin{aligned} &r^1_{1j} + V_t(x+e_1) - \max\left\{r^1_{1j} + V_t(x+2e_1), r^2_{2j} + V_t(x+e_1+e_2), V_t(x+e_1)\right\} \\ &\quad \leq \max\left\{r^1_{1j} + V_t x+e_1+e_2), r^2_{2j} + V_t(x+2e_2), V_t(x+e_2)\right\} \\ &\qquad - \max\left\{r^1_{1j} + V_t(x+2e_1+e_2), r^2_{2j} + V_t(x+e_1+2e_2), V_t(x+e_1+e_2)\right\}. \end{aligned}$$

The remaining 6 cases can be examined analogously. Hence, we obtain inequality (13).

Second, we proof subconcavity by showing that $T_{3j}(V_t)(x+e_1) - T_{3j}(V_t)(x+e_2)$ is nonincreasing in x_2, or equivalently,

$$\begin{aligned} &\max\left\{r^1_{1j} + V_t(x+2e_1), r^2_{2j} + V_t(x_1+e_1+e_2), V_t(x_1+e_1)\right\} \\ &\quad - \max\left\{r^1_{1j} + V_t(x+e_1+e_2), r^2_{2j} + V_t(x+2e_2), V_t(x+e_2)\right\} \\ &\geq \max\left\{r^1_{1j} + V_t(x+3e_1), r^2_{2j} + V_t(x+2e_1+e_2), V_t(x+2e_1)\right\} \\ &\quad - \max\left\{r^1_{1j} + V_t(x+2e_1+e_2), r^2_{2j} + V_t(x+e_1+2e_2), V_t(x+e_1+e_2)\right\}. \end{aligned} \tag{14}$$

To verify (14), we consider the first 3 cases:

$$\begin{aligned} &V_t(x+2e_1) - V_t(x+e_1+e_2) \\ &\quad \geq V_t(x+3e_1) - V_t(x+2e_1+e_2) \\ &V_t(x+e_1+e_2) - V_t(x+e_1+e_2) \\ &\quad = V_t(x+2e_1+e_2) - V_t(x+2e_1+e_2) \\ &V_t(x+e_1) - V_t(x+e_1+e_2) \\ &\quad \geq V_t(x+2e_1) - V_t(x+2e_1+e_2). \end{aligned}$$

Solving the remaining 6 cases similarly, it follows that T_{3j} is subconcave. Hence, T_{3j} is multimodular. This completes the proof. □

Lemma 12. *For models RMFP-s and RMFP-d, the value function is submodular with respect to time, i.e.,*

(i) $\Delta_1 V_t(x) \geq \Delta_1 V_{t-1}(x) \quad \forall x \in \mathcal{X}_t, t \in \mathcal{T}$
(ii) $\Delta_2 V_t(x) \geq \Delta_2 V_{t-1}(x) \quad \forall x \in \mathcal{X}_t, t \in \mathcal{T}$.

Proof. We show the result for RMFP-s. Define the difference operators $\Delta^-_i V_t(x) = V_t(x+e_i) - V_t(x)$ and $\Delta^-_i H_{kj}(V_t,d)(x) = H_{kj}(V_t,d)(x+e_i) - H_{kj}(V_t,d)(x)$ for $i \in \{1,2\}$. To prove part (i) of this lemma, it is equivalent to show that $\Delta^-_1 V_t(x)$ is non*in*creasing in t, i.e., $\Delta^-_1 V_t(x) - \Delta^-_1 V_{t-1}(x) \leq 0$ for all $t \in \mathcal{T}$.

First, let us consider a given period t in which any flexible product is scheduled to arrive, i.e., $t \in \{t \in \mathcal{T} : k(t) = 3 \text{ and } j(t) = j \in \{1,\dots,f\}\}$. Using our Lemma 5, we can rewrite the Bellman equation

$$\begin{aligned} V_t(x) = {} & p_{3j0t}\, H_{3j}(V_{t-1},0)(x) \\ &+ \sum_{d\geq 1} p_{3jdt} \max\{r^1_{3j} + \Delta^-_1 H_{3j}(V_t,d-1)(x), r^2_{3j} + \Delta^-_2 H_{3j}(V_t,d-1)(x), 0\} \\ &+ \sum_{d\geq 1} p_{3jdt}\, H_{3j}(V_t,d-1)(x). \end{aligned} \tag{15}$$

Hence, we analyze the difference

$$\begin{aligned} &\Delta^-_1 V_t(x) - \Delta^-_1 V_{t-1}(x) \\ &\quad = V_t(x+e_1) - V_t(x) - \Delta^-_1 H_{3j}(V_{t-1},0)(x) \\ &\quad = (p_{3j0t} - 1)\Delta^-_1 H_{3j}(V_{t-1},0)(x) + \sum_{d\geq 1} p_{3jdt}\Delta^-_1 H_{3j}(V_{t-1},d-1)(x) \\ &\quad + \sum_{d\geq 1} p_{3jdt}\big(\max\{r^1_{3j} + \Delta^-_1 H_{3j}(V_{t-1},d-1)(x+e_1), r^2_{3j} + \Delta^-_2 H_{3j}(V_{t-1},d-1)(x+e_1), 0\} \\ &\qquad\qquad - \max\{r^1_{3j} + \Delta^-_1 H_{3j}(V_{t-1},d-1)(x), r^2_{3j} + \Delta^-_2 H_{3j}(V_{t-1},d-1)(x), 0\}\big) \end{aligned} \tag{16}$$

where the first equality follows from the identity $\Delta^-_1 V_{t-1}(x) = \Delta^-_1 H_{3j}(V_{t-1},d)(x)$ in the case $d = 0$, whereas the second equality is the result of inserting the defintion of optimality (15).

Next, we show that the sum of the first and the second term on the right hand side of (16) is nonpositive. To achieve this, these terms can, by (1), be equivalently rewritten:

$$
\begin{aligned}
-\sum_{d\geq 1} & p_{3jdt}\Delta_1^- H_{3j}(V_{t-1},0)(x) + \sum_{d\geq 1} p_{3jdt}\Delta_1^- H_{3j}(V_{t-1},d-1)(x) \\
&= \sum_{d\geq 1} p_{3jdt}\Big(H_{3j}(V_{t-1},d-1)(x+e_1) - H_{3j}(V_{t-1},0)(x+e_1) \\
&\qquad - H_{3j}(V_{t-1},d-1)(x) + H_{3j}(V_{t-1},0)(x)\Big) \\
&= \sum_{d\geq 1} p_{3jdt}\Bigg(\sum_{z=1}^{d-1} \Big(H_{3j}(V_{t-1},z)(x+e_1) - H_{3j}(V_{t-1},z-1)(x+e_1) \\
&\qquad - H_{3j}(V_{t-1},z)(x) + H_{3j}(V_{t-1},z-1)(x)\Big)\Bigg) \\
&= \sum_{d\geq 1} p_{3jdt}\Bigg(\sum_{z=1}^{d-1} \Big(\Delta_1^- H_{3j}(V_{t-1},z)(x) - \Delta_1^- H_{3j}(V_{t-1},z-1)(x)\Big)\Bigg). \qquad (17)
\end{aligned}
$$

Note that the first and third equalities hold by rearranging terms, and the second equality holds by using

$$
H_{kj}(V_t,d)(x) - H_{kj}(V_t,0)(x) = \sum_{z=1}^{d} \Big(H_{kj}(V_t,z)(x) - H_{kj}(V_t,z-1)(x)\Big),
$$

where we take the value of the above summation to be zero if $d = 0$. Thus, to show that the inner summation on the right hand side of (17) is nonpositive, we need to prove that $\Delta_1^- H_{3j}(V_{t-1},d)(x) \leq \Delta_1^- H_{3j}(V_{t-1},d-1)(x)$, or equivalently,

$$
\begin{aligned}
H_{3j}(V_{t-1},d)(x+e_1) &- H_{3j}(V_{t-1},d-1)(x+e_1) \\
&\leq H_{3j}(V_{t-1},d)(x) - H_{3j}(V_{t-1},d-1)(x)
\end{aligned} \qquad (18)
$$

for all $d \in \{1,2,\ldots,\}$. The proof is by induction over the demand levels. Using the definition of $H_{3j}(V_t,d)(x)$, we obtain from (18)

$$
\begin{aligned}
\max\{r_{3j}^1 &+ \Delta_1^- H_{3j}(V_{t-1},d-1)(x+e_1), r_{3j}^2 + \Delta_2^- H_{3j}(V_{t-1},d-1)(x+e_1), 0\} \\
&\leq \max\{r_{3j}^1 + \Delta_1^- H_{3j}(V_{t-1},d-1)(x), r_{3j}^2 + \Delta_2^- H_{3j}(V_{t-1},d-1)(x), 0\}.
\end{aligned} \qquad (19)
$$

Now, recall that $H_{3j}(V_t,d)(x)$ is, by Lemma 6, componentwise concave and submodular for all possible demand realizations at any given time $t \in \mathcal{T}$. With the difference operators Δ_i^- these properties are translated into the inequalities

$$
\Delta_1^- H_{3j}(V_t,d)(x+e_1) \leq \Delta_1^- H_{3j}(V_t,d)(x) \qquad \text{and} \qquad \Delta_2^- H_{3j}(V_t,d)(x+e_1) \leq \Delta_2^- H_{3j}(V_t,d)(x),
$$

which hold for all $x \in \mathcal{X}_t$ and $d \in \{0,1,\ldots\}$. Hence, inequality (18) follows from the repeated application of the above monotonicity statements to inequality (19) for all $d \in \{1,2,\ldots\}$.

Finally, it can be easily seen that the last term on the right hand side of equation (16) is nonpositive because of the verification of inequality (19) for all $d \in \{1,2,\ldots\}$.

In all remaining cases, i.e., for all $t \in \{t \in \mathcal{T} : k(t) \in \{1,2\}$ and $j(t) = j \in \{1,\ldots,f\}\}$, it is straightforward and hence omitted to show that $\Delta_1^- V_t(x) - \Delta_1^- V_{t-1}(x) \leq 0$ by the same chain of arguments. Herewith, the result of part (i) is proved. Part (ii) of this lemma is true by symmetry.

For model RMFP-d, the both results of this lemma were shown by Chen *et al.* (2010). □

B. Algorithms

Algorithm 2: Computation of switching curve $S_{3j,t-1}$

Input : j, t and $V_{t-1}(x)$ for all $x \in \mathcal{X}_{t-1}$

Compute $S_{3j,t-1}(0)$ and let $\nu_1 := S_{3j,t-1}(0)$

for $x_2 = 1, \ldots, c_2$ **do**

  while $\min\{\Delta_1 V_{t-1}(\nu_1, x_2) - r_{3j}^1, \Delta_2 V_{t-1}(\nu_1, x_2) - r_{3j}^2\} \geq 0$ **and** $\nu_1 \geq 0$ **do**

    $\nu_1 := \nu_1 - 1$

  $S_{3j,t-1}(x_2) := \nu_1 + 1$

Output: $\{S_{3j,t-1}(x_2)$ for all $x_2 \in \{0, \ldots, c_2\}\}$
